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Canonical Modules of Partially Ordered Sets

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In this brief note we show that the canonical module of the Stanley–Reisner ring of a doubly Cohen–Macaulay ordered set is isomorphic to a certain ideal of the same ring. For a general finite partially ordered set, the corresponding ideal is isomorphic to a submodule of the canonical module. For an introduction to Cohen–Macaulay ordered sets from the ring-theoretical point of view see Garsia [4] and Baclawski–Garsia [3]. Doubly Cohen–Macaulay ordered sets were introduced in Baclawski [2].

Let Δ be a finite simplicial complex of rank r (dimension $r-1$) on vertex set V. We write Δ_k for $\{\sigma \in \Delta \mid |\sigma| = k\}$. We assume that Δ can be *colored*, i.e., there exists a map $c\colon V \to [r] = \{1, 2,..., r\}$ such that for every $\sigma \in \Delta$, $|c(\sigma)| = |\sigma|$; and we fix a choice of coloring henceforth. The most important example of a colored complex is the simplicial complex $\Delta(P)$ of chains of a partially ordered set (poset) P. For example, the map $c\colon P \to [r]$ given by

$$c(x) = \max\{k \mid \text{there exists } x_1,..., x_{k-1} \in P \text{ such that } x_1 < x_2 < \cdots < x_{k-1} < x\},$$

is a coloring of $\Delta(P)$.

If we regard each element of V as an indeterminate, then the *Stanley–Reisner ring* is the quotient ring

$$K[\Delta] = K[v \mid v \in V]/(v_1 \cdots v_k \mid \{v_1,..., v_k\} \notin \Delta).$$

Let $\theta_i \in K[\Delta]$ be defined by

$$\theta_i = \sum_{v \in V} v\chi(c(v) = i),$$

where $\chi(\mathscr{P})$ is 1 if $\mathscr{P}$ is true and 0, otherwise. We call θ_i the ith *rank sum* of $K[\Delta]$. The sequence $\theta_1,..., \theta_r$ is a homogeneous system of parameters (*frame*)

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of $K[\Delta]$. We write $K[\theta]$ for $K[\theta_1,\dots,\theta_r] \subseteq K[\Delta]$. The simplicial complex Δ is said to be *Cohen–Macaulay* (CM) if $K[\Delta]$ is a free $K[\theta]$ module.

A simplicial complex Δ is said to be 2-CM *connected* (or *doubly* CM) if for every vertex $v \in V$, we have

$$\Delta\backslash\{v\} = \{\sigma \in \Delta \mid v \notin \sigma\} \qquad \text{is CM}, \tag{1}$$

$$r(\Delta\backslash\{v\}) = r(\Delta). \tag{2}$$

An important example of such a complex is $\Delta(P)$, where $P = L\backslash\{\hat{0}, \hat{1}\}$ and L is a finite geometric lattice. For a proof of this and for other examples see [2].

The *canonical module* of a CM graded K algebra R is the graded R module

$$\Omega(R) = \operatorname{Hom}_{K[\alpha]}(R, K[\alpha]),$$

where $\alpha_1,\dots,\alpha_r$ is any frame of R. The canonical module is independent of the choice of a frame, see Herzog–Kunz [5, Remark 5.19]. We will use this as the definition of $\Omega(R)$ even when R is not CM.

We will use multiindex notation for denoting nonzero monomials of $K[\Delta]$. Thus if $N = (n_1,\dots,n_k)$ is a sequence of nonnegative integers, then v^N stands for the monomial $v^N = \prod_{i=1}^k (v_i)^{n_i}$. We will always assume that the vertices $v_1,\dots,v_k$ in such a monomial satisfy $c(v_1) < c(v_2) < \cdots < c(v_k)$. The *rank set* $r(v^N)$ is the *multiset* for which each color $c(v_i)$ occurs exactly as often as v_i occurs as a factor of v^N. Equivalently we may think of $r(v^N)$ as the r-tuple $\sum_{i=1}^k n_i e(c(v_i))$, where $e(j)$ is the r-tuple whose ith component is δ_{ij}. The vector ρ is the r-tuple $(1,\dots,1) = \sum_{i=1}^r e(i)$. The rank set defines a *multigrading* on $K[\Delta]$. The *support*, $\operatorname{supp}(v^N)$, of the monomial v^N is the set $\{v_i \mid n_i > 0\}$. Recall that an (*algebraic*) *chain* is a formal linear combination of elements of Δ. For each $\sigma \in \Delta$ there is a corresponding monomial $\prod \sigma = \prod_{v\in\sigma} v \in K[\Delta]$. Thus we may regard the space of chains as a subspace of $K[\Delta]$. In particular the $(r-1)$th reduced homology $\tilde{H}_{r-1}(\Delta, K)$, being a subspace of the space of chains of rank r, is a subspace of $K[\Delta]$. Write $J(\Delta)$ for the ideal of $K[\Delta]$ generated by $\tilde{H}_{r-1}(\Delta, K)$.

Define a linear map $\varphi\colon K[\Delta] \to K[\theta]$ as

$$\varphi(v^N) = \theta_1^{n_1-2}\cdots\theta_r^{n_r-2}, \qquad \text{if} \quad r(v^N) \geqslant 2\rho,$$
$$= 0, \qquad \text{otherwise}.$$

The inequality $r(v^N) \geqslant 2\rho$ is componentwise. Note that $k = r$ when this inequality holds. Now φ is a multihomogeneous linear map of degree $-2r$, which is generally not a $K[\theta]$ homomorphism. We use the map φ to define a homomorphism

$$\psi: J(\Delta) \to \mathrm{Hom}_K(K[\Delta], K[\theta]),$$

as follows: if $g \in J(\Delta)$ and $f \in K[\Delta]$, then let $\psi(g)(f)$ be $\varphi(gf)$. It is easy to see that ψ is a $K[\Delta]$ homomorphism. We first show that ψ is injective; no special property of $J(\Delta)$ is used for this. Next we prove that $\mathrm{Im}\,\psi$ is contained in $\Omega(K[\Delta])$. Finally, we prove that $\mathrm{Im}\,\psi = \Omega(K[\Delta])$ when Δ is doubly CM.

LEMMA. *ψ is injective.*

Proof. Suppose that $\psi(g) = 0$. We may assume that g is multihomogeneous. Note that every term of g is supported on a simplex $\sigma \in \Delta_r$. Let av^N be one term of g. Then $0 = \psi(g)(v^\rho) = \varphi(gv^\rho) = \varphi(av^{N+\rho}) = a\theta^{N-\rho}$. Thus $a = 0$. We may do the same for any term of g. Hence $g = 0$.

Q.E.D.

We now come to our first main result.

THEOREM 1. *For any colored complex Δ, there is a monomorphism*

$$\psi: J(\Delta) \to \Omega(K[\Delta]).$$

Proof. We wish to show that $\mathrm{Im}\,\psi \subseteq \mathrm{Hom}_{K[\theta]}(K[\Delta], K[\theta])$. Clearly it suffices to show that $\varphi(\theta_i g) = \theta_i \varphi(g)$ for any i and any $g \in J(\Delta)$. Moreover we may assume that g is multihomogeneous.

Consider first the case $r(g) \geqslant 2\rho$. If g were simply a monomial v^N, then $\theta_i g$ would be $v^{N+e(i)}$ and hence $\varphi(\theta_i g) = \theta^{N+e(i)-2\rho} = \theta_i \varphi(g)$. In general g would be a linear combination of such terms, so the result follows by linearity. Next suppose that $e(i) + r(g) \not\geqslant 2\rho$. Then $\varphi(\theta_i g) = 0$ and $\varphi(g) = 0$ so this case also follows.

There remains the case for which $r(g) \not\geqslant 2\rho$ but $e(i) + r(g) \geqslant 2\rho$. Each term of g has the form av^N, where $N = (n_1, \dots, n_r)$ satisfies $n_i = 1$ and $n_j \geqslant 2$ for $j \neq i$. By definition of $J(\Delta)$ we may find $g_j \in K[\Delta]$ and $f_j \in \tilde{H}_{r-1}(\Delta, K)$ such that $g = \sum_j g_j f_j$. Now $r(f_j) = \rho$ so the support of every term of g is a simplex in Δ_{r-1}. Thus we may write

$$f_j = \sum_{\sigma \in \Delta_r} c_j(\sigma) \prod \sigma, \qquad g_j = \sum_{\tau \in \Delta_{r-1}} d_j(\tau)\, h(\tau),$$

where $c_j(\sigma)$ and $d_j(\tau)$ are in K, and where $h(\tau)$ is the unique monomial such that $r(h(\tau)) = r(g_j)$ and $\mathrm{supp}(h(\tau)) = \tau$. We can then compute

$$g = \sum_j g_j f_j = \sum_j \sum_{\tau \in \Delta_{r-1}} \sum_{\sigma \in \Delta_r} d_j(\tau)\, c_j(\sigma)\, h(\tau) \prod \sigma.$$

Now $h(\tau)\prod\sigma=0$ unless $\tau\subset\sigma$. Thus

$$g=\sum_j\sum_{\sigma,\tau} c_j(\sigma)\, d_j(\tau)\, h(\tau)\prod\sigma\chi(\sigma\supset\tau).$$

Next we use the fact that $f_j\in\tilde{H}_{r-1}(\Delta, K)=\operatorname{Ker}(\partial)$. Thus

$$\begin{aligned} 0=\partial f_j &= \partial\sum_\sigma c_j(\sigma)\prod\sigma \\ &= \sum_\sigma c_j(\sigma)\sum_{\tau\in\Delta_{r-1}}(-1)^{c(\sigma\backslash\tau)}\prod\tau\chi(\tau\subset\sigma) \\ &= \sum_{\tau\in\Delta_{r-1}}\left(\sum_{\sigma\supset\tau}(-1)^{c(\sigma\backslash\tau)}c_j(\sigma)\right)\prod\tau. \end{aligned}$$

Since $c(\sigma\backslash\tau)=r-c(\tau)$ depends only on τ, it follows that for every $\tau\in\Delta_{r-1}$, we have $\sum_{\sigma\supset\tau}c_j(\sigma)=0$. Now $\varphi(\theta_i h(\tau)\prod\sigma)=\theta^{r(g)+e(i)-2\rho}$ is the same for every $\sigma\in\Delta_r$ and $\tau\in\Delta_{r-1}$. Thus

$$\varphi(\theta_i\, g)=\sum_j\sum_\tau\left(\sum_{\sigma\supset\tau}c_j(\sigma)\right)d_j(\tau)\,\theta^{r(g)+e(i)-2\rho}=0.$$

Since $\varphi(g)=0$, it follows that $\varphi(\theta_i\, g)=\theta_i\varphi(g)$, so φ is a $K[\theta]$ homomorphism in all cases. Q.E.D.

We now invoke a characterization of the 2-CM property in terms of resolutions in order to obtain

THEOREM 2. *Let Δ be a colored,* CM *complex. Then Δ is* 2 CM *if and only if $\psi\colon J(\Delta)\to\Omega(K[\Delta])$ is an isomorphism.*

Proof. Since Δ is CM, $|\mu(\Delta)|=\tilde{h}_{r-1}(\Delta, K)=\dim_K\tilde{H}_{r-1}(\Delta, K)$. Now if ψ is an isomorphism, then $\Omega(K[\Delta])$ is generated by $|\mu(\Delta)|$ elements. By [2, Corollary 4.7], Δ is 2 CM.

Conversely, in the proof of [2, Theorem 4.5] it is shown that if Δ is 2 CM, then the minimum number of generators of $\Omega(K[\Delta])$ is $|\mu(\Delta)|$, and moreover, all of the generators in a minimum set have the same degree. See also the discussion in [1, Sect. 7]. Since $\Omega(K[\Delta])\neq 0$ this implies that $\mu(P)\neq 0$ and hence that $\tilde{H}_{r-1}(\Delta, K)\neq 0$. By standard results in commutative algebra (see [3, Proposition 2.3(3)]), $K[\Delta]$ is a free $K[\theta]$ module with a basis given by any linear basis of $K[\Delta]/(\theta)$. By [3, Theorem 5.1], the highest degree of a homogeneous element of $K[\Delta]/(\theta)$ is *exactly* r, since $\tilde{H}_{r-1}(\Delta, K)\neq 0$. Thus the lowest degree of an element of $\Omega(K[\Delta])$ is $-r$. Now ψ has degree $-2r$ so the generators of $J(\Delta)$ get mapped to elements of $\Omega(K[\Delta])$ of degree $-r$. Since ψ is injective by Theorem 1, and since $\Omega(K[\Delta])$ has $|\mu(\Delta)|=\tilde{h}_{r-1}(\Delta, K)$ generators all of the same degree, it follows that ψ is also surjective. Q.E.D.

Remark. It is clear that φ allows us to define an injective map $\psi: K[\Delta] \to \operatorname{Hom}_K(K[\Delta], K[\theta])$ for any pure, colorable, simplicial complex. Thus $\Omega(K[\Delta])$ is isomorphic to an ideal of $K[\Delta]$ whenever Im ψ contains $\Omega(K[\Delta])$. It would be interesting to find conditions under which this holds.

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