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Fixed Points in Partially Ordered Sets

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INTRODUCTION

In this paper we present a number of theorems about fixed points of mappings of partially ordered sets. Our approach is based on a discrete form of the Hopf-Lefschetz fixed point theorem and on order-theoretical analogs of topological constructions. However, we show by example that the fixed point theory of partially ordered sets cannot be reduced to topological fixed point theory. Nevertheless, a substantial number of previously known results in this field are not only subsumed under our approach but are also extended and refined. This is particularly true in the finite case where certain qualitative properties of the fixed point sets come within reach which are stronger than that of merely being nonvoid. We also show that, somewhat surprisingly, fixed point theory has applications to the question of the existence of complements in finite lattices.

Let P be a poset (partially ordered set). A *self-map* is a function $f: P \to P$. The *fixed point set* of f is the subposet $P^f = \{x \in P \mid x = f(x)\}$. A theorem of A. Tarski [14] states that if P is a complete lattice and if f is an order-preserving self-map then P^f is nonempty and forms a complete lattice under the inherited order. Posets P such that the fixed point set is nonempty for any order-preserving self-map are said to have the *fixed point property*. For lattices the fixed point property is equivalent to lattice completeness (Tarski [14] and Davis [7]). The fixed point property for posets in general turns out to be much more subtle than the lattice case would lead one to expect.

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This paper is organized as follows. In Section 1 we derive our basic tool, the Hopf-Lefschetz fixed point theorem for order-preserving and for order-reversing self-maps of a finite poset. A corollary of this theorem is that if a finite poset P is *acyclic* (i.e., if P has the homology of a point) and if f is order-preserving, then P^f is nonempty and has the Euler characteristic of a point. We apply this in Sections 2 and 3 to show that some well-known classes of posets have the fixed point property. Most of these results appear to be new. Section 2 also contains a number of examples to illustrate the relationships among the various concepts we have introduced and to show that the fixed point theory of posets cannot be reduced to topological fixed point theory.

In Section 3 we discuss the relationship between complements in lattices and the existence of fixed points of self-maps. We prove, using fixed point theory, that finite lattices having an order-reversing self-map of a certain kind are *complemented* lattices (i.e., every element of the lattice has a complement).

In the last two sections of the paper we consider whether the fixed point property is preserved under various poset constructions. Our results are valid for infinite as well as finite posets. In Section 4, for example, we completely characterize those well-ordered-complete posets P such that the cardinal power P^P has the fixed point property. In Section 5 we consider the direct product $P \times Q$ of posets and also a poset analogue of the topological method of "gluing" together locally defined spaces. We show that under certain conditions these two constructions preserve the fixed point property.

The authors would like to thank Garrett Birkhoff for teaching us about lattices and posets and for always stressing the connections between lattice theory and other parts of mathematics. We also want to thank Felix Browder, Ivan Rival, Gian-Carlo Rota and Richard Stanley for their advice and encouragement.

1. The Hopf-Lefschetz Fixed Point Theorem

We begin with some definitions and notation from the theory of partially ordered sets and from algebraic topology, which will be of use throughout the paper. For a more detailed account of the order-theoretic and topological concepts we use, see Birkhoff [4] and Brown [6], respectively.

Let P be a poset. Then $\hat{P}$ is the poset obtained by adjoining two new elements $\hat{0}$ and $\hat{1}$ to P such that $\hat{0} < x < \hat{1}$ in $\hat{P}$ for all $x \in P$. If we specify that a poset Q has a minimum and a maximum, e.g., when Q is a finite lattice, we will denote them by $\hat{0}$ and $\hat{1}$ respectively. Thus, statements such as $Q = \hat{P}$ have the obvious meaning.

A map $f: P \to Q$ is *order-preserving* if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in P$. Given any map $f: P \to P$ we write P^f for the *fixed point set*

$$\{x \in P \mid x = f(x)\} \text{ of } f.$$

The *order complex* of a poset P is the simplicial complex $\Delta(P)$ whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_n$ of P. The partial order on the vertices of $\Delta(P)$ defines an orientation on each simplex of $\Delta(P)$. The *geometric realization* of P, denoted by $|P|$ or $|\Delta(P)|$, is the polyhedron associated to the simplicial complex $\Delta(P)$, as usually defined in topology. An order-preserving map $f: P \to P$ induces a simplicial map $\Delta(f): \Delta(P) \to \Delta(P)$ which clearly is orientation-preserving.

Let K be a field. Let $C_*(P, K)$ be the algebraic chain complex over $\Delta(P)$ with coefficients in K. Write $d_n: C_n(P, K) \to C_{n-1}(P, K)$ for the differential (boundary map) of the complex $C_*(P, K)$. Then, as is usual, we write $B_n = \mathrm{Im}(d_{n+1})$, $Z_n = \mathrm{Ker}(d_n)$ and $H_n(P, K) = Z_n/B_n$. The K-vector spaces $H_n(P, K)$ are the *homology groups* of P *with coefficients in* K.

In the remainder of this paper we shall, for simplicity, use only homology with rational coefficients, although most of the results hold with obvious modifications over any field. We write $\mathbb{Q}$ for the field of rational numbers.

Let P be a finite poset. An *i-chain* is a chain $x_0 < x_1 < \cdots < x_i$ with $i + 1$ elements (i.e., an i-simplex of $\Delta(P)$). The *length* of P is the integer $\ell(P)$ such that P contains an ℓ-chain but no $(\ell + 1)$-chain (i.e., $\ell(P) = \dim(\Delta(P))$). Let $\alpha_i(P)$ be the number of i-chains of P. The *Euler characteristic* $\chi(P)$ is defined by $\chi(P) = \sum_{i=0}^{\infty} (-1)^i \alpha_i(P)$; in particular, $\chi(\varnothing)$ is zero. The *Möbius function* $\mu(P)$ of P is defined as the value of $\mu(\hat{0}, \hat{1})$ computed in $\hat{P}$ (readers unfamiliar with μ should consult Rota [13]). It is a theorem of P. Hall's that $\chi(P) = \mu(P) + 1$ (Rota [13, Prop. 6, p. 346]). One may also compute $\chi(P)$ using the homology of P. The well-known Euler-Poincaré formula (Brown [6, p. 10]) states that $\chi(P)$ equals the alternating sum of the dimensions of the $H_n(P, \mathbb{Q})$:

$$\chi(P) = \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{Q}} H_n(P, \mathbb{Q}).$$

Again, let P be finite. For an order-preserving map $f: P \to P$ let $f_n: H_n(P, \mathbb{Q}) \to H_n(P, \mathbb{Q})$ be the linear map which is functorially induced on homology. The *Lefschetz number* of f is

$$\Lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(f_n),$$

where $\operatorname{Tr}(f_n)$ denotes the trace of the linear map f_n.

We are now ready to formulate and prove two versions of the Hopf-Lefschetz fixed point theorem.

THEOREM 1.1. *Let P be a finite poset and let $f: P \to P$ be an order-preserving map. Then*

$$\Lambda(f) = \chi(P^f).$$

In particular, if $\Lambda(f) \neq 0$, then $P^f \neq \varnothing$.

This is not the same as the usual simplicial version of the Hopf-Lefschetz theorem: for a general simplicial map $g: \Delta(P) \to \Delta(P)$ the Lefschetz number $\Lambda(g)$ need not have any relationship to P^g.

Proof. f induces maps $\hat{f}_n: C_n(P, \mathbb{Q}) \to C_n(P, \mathbb{Q})$ which together form a map $\hat{f}_*: C_*(P, \mathbb{Q}) \to C_*(P, \mathbb{Q})$ of chain complexes. Let $T(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(\hat{f}_n)$. We first compute $T(f)$ using the ordinary basis of $C_*(P, \mathbb{Q})$:

$$\begin{aligned} T(f) &= \sum_{n=0}^{\infty} (-1)^n \#\{(x_0 < x_1 < \cdots < x_n) \mid \{f(x_i)\} = \{x_i\}\} \\ &= \sum_{n=0}^{\infty} (-1)^n \#\{(x_0 < x_1 < \cdots < x_n) \mid f(x_i) = x_i \text{ for all } i\} \\ &= \chi(P^f), \end{aligned}$$

because $\{(x_0 < x_1 < \cdots < x_n) \mid f(x_i) = x_i \text{ for all } i\}$ is the set of n-simplices of $\Delta(P^f)$. Note that in general a simplicial map can fix a simplex without fixing any of its vertices. In our case this cannot occur since f is order-preserving.

We now compute $T(f)$ using another basis. The fact that $\hat{f}_*$ commutes with d_* implies that $\hat{f}_n$ induces maps

$$\begin{aligned} \hat{f}_n^{(0)}&: B_n \to B_n \\ \hat{f}_n^{(1)}&: Z_n/B_n \to Z_n/B_n \\ \hat{f}_n^{(2)}&: C_n/Z_n \to C_n/Z_n . \end{aligned}$$

Now, by definition of Z_n and B_n, $d_n: C_n/Z_n \to B_{n-1}$ is an isomorphism. Again because $\hat{f}_*$ commutes with d_*, this diagram commutes:

$$\begin{tikzcd} C_n/Z_n \arrow[r, "d_n", "\sim"'] \arrow[d, "\hat{f}_n^{(2)}"] & B_{n-1} \arrow[d, "\hat{f}_{n-1}^{(0)}"] \\ C_n/Z_n \arrow[r, "\sim", "d_n"'] & B_{n-1} \end{tikzcd}$$

Therefore, $\operatorname{Tr}(\hat{f}_n^{(2)}) = \operatorname{Tr}(\hat{f}_{n-1}^{(0)})$. But then

$$\begin{aligned} \operatorname{Tr}(\hat{f}_n) &= \operatorname{Tr}(\hat{f}_n^{(0)}) + \operatorname{Tr}(\hat{f}_n^{(1)}) + \operatorname{Tr}(\hat{f}_n^{(2)}) \\ &= \operatorname{Tr}(\hat{f}_n^{(0)}) + \operatorname{Tr}(\hat{f}_n^{(1)}) + \operatorname{Tr}(\hat{f}_{n-1}^{(0)}). \end{aligned}$$

The first equality above is easily seen by computing $\operatorname{Tr}(\hat{f}_n)$ in a basis for C_n assembled by first choosing a basis for B_n, then extending to Z_n and finally extending to a basis for all of C_n. Hence, taking the alternating sum we find that

$$T(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(\hat{f}_n^{(1)}) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(f_n) = \Lambda(f). \quad \blacksquare$$

Now suppose that $f: P \to P$ is *order-reversing*, i.e., that $x \leqslant y$ implies $f(x) \geqslant f(y)$ for all $x, y \in P$. Let $P_f = \{x \in P \mid x = f^2(x) \leqslant f(x)\}$. Thus, the fixed point set P^f is a subset of P_f. Since f is order-reversing it gives rise to a simplicial map $\Delta(f): \Delta(P) \to \Delta(P)$ and hence to maps of chain complexes $\hat{f}_*: C_*(P, \mathbb{Q}) \to C_*(P, \mathbb{Q})$ and of homology $f_n: H_n(P, \mathbb{Q}) \to H_n(P, \mathbb{Q})$. Thus the Lefschetz number $\Lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(f_n)$ is again well-defined.

THEOREM 1.2. *Let P be a finite poset and let $f: P \to P$ be an order-reversing map. Then*

$$\Lambda(f) = \chi(P_f).$$

Consequently, if there are no elements x of P such that $x = f^2(x) < f(x)$, then f has precisely $\Lambda(f)$ fixed points.

Proof. With suitable modifications the same proof we used for Theorem 1.1 goes through. Let $T(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(\hat{f}_n)$. Just as before we find that $T(f) = \Lambda(f)$. However, to compute $T(f)$ in the ordinary basis of $C_*(P, \mathbb{Q})$ we have to take into account the fact that $\Delta(f)$ is not orientation-preserving. $\Delta(f)$ preserves the orientation of an i-simplex if $i \equiv 0$ or $3 \pmod 4$ and reverses it if $i \equiv 1$ or $2 \pmod 4$. Hence, if we define ϵ_n to be $+1$ if $n \equiv 0$ or $3 \pmod 4$ and -1 if $n \equiv 1$ or $2 \pmod 4$, then

$$\begin{aligned}
T(f) &= \sum_{n=0}^{\infty} (-1)^n \epsilon_n \#\{(x_0 < x_1 < \cdots < x_n) \mid \{f(x_i)\} = \{x_i\}\} \\
&= \sum_{m=0}^{\infty} (-1)^m \#\{(x_0 < x_1 < \cdots < x_{2m}) \mid \{f(x_i)\} = \{x_i\}\} \\
&\quad + \sum_{m=0}^{\infty} (-1)^m \#\{(x_0 < x_1 < \cdots < x_{2m+1}) \mid \{f(x_i)\} = \{x_i\}\} \\
&= \sum_{m=0}^{\infty} (-1)^m [\#\{(x_0 < x_1 < \cdots < x_{2m}) \mid f(x_i) = x_{2m-i} \text{ for all } i\} \\
&\quad + \#\{(x_0 < x_1 < \cdots < x_{2m+1}) \mid f(x_i) = x_{2m+1-i} \text{ for all } i\}] \\
&= \chi(P_f),
\end{aligned}$$

because there is an obvious bijection between the set of all m-simplices of $\Delta(P_f)$ and the set

$$\{(x_0 < x_1 < \cdots < x_n) \mid n = 2m \text{ or } 2m+1 \text{ and } f(x_i) = x_{n-i} \text{ for all } i\}.$$

If there are no elements satisfying $x = f^2(x) < f(x)$ in P then $P^f = P_f$. Two distinct fixed points of an order-reversing map cannot be comparable so $\Lambda(f)$ is the Euler characteristic of the zero-dimensional complex $\Delta(P^f)$, and hence $\Lambda(f) = \# P^f$. ∎

EXAMPLE 1.3. It is instructive to consider the case of the Boolean algebra B_n of subsets of the set $\{1,..., n\}$. Let $\hat{P} = B_n$.

Begin with the order-preserving case. If $f: P \to P$ is surjective, then f is an automorphism of P so it has the form $f = f_\pi$, where π is a permutation of $\{1,..., n\}$ and f_π is the natural extension of π to P. $\Delta(P)$ is the first barycentric subdivision of the boundary of an $(n-1)$-simplex; hence $|P|$ is homeomorphic to the $(n-2)$-dimensional sphere S^{n-2}. Therefore P has two nonvanishing homology groups: $H_0(P)$ and $H_{n-2}(P)$, both isomorphic to $\mathbb{Q}$. It is easy to see that $\operatorname{Tr}(f_0) = 1$ and $\operatorname{Tr}(f_{n-2}) = \operatorname{sgn}(\pi)$. Therefore $\Lambda(f) = 1 + (-1)^{n-2} \operatorname{sgn}(\pi)$. The fixed points of f are unions of orbits of π and so $\widehat{(P^f)}$ is itself a Boolean algebra. If π has just one cycle, P^f is empty. Let the cycles of π have lengths $n_1, n_2,..., n_k$. Then $\chi(P^f) = 1 + (-1)^{k-2}$, even when $k = 1$. On the other hand, $\operatorname{sgn}(\pi) = (-1)^{n_1-1} \cdots (-1)^{n_k-1} = (-1)^{n_1+\cdots+n_k-k} = (-1)^{n-k}$. Thus $\Lambda(f) = 1 + (-1)^{n-2} \operatorname{sgn}(\pi) = 1 + (-1)^{n-2}(-1)^{n-k} = 1 + (-1)^{k-2} = \chi(P^f)$.

Now suppose that $f: P \to P$ is not surjective. Then the form of f cannot be so easily characterized. However, a generating cycle of $H_{n-2}(P, \mathbb{Q})$ cannot be mapped by f_{n-2} to a nonzero multiple of itself. Hence, in this case, $\Lambda(f) = \operatorname{Tr}(f_0) = 1$, and f has a fixed point.

We now consider the order-reversing case. If $f: P \to P$ is surjective, then f is an anti-automorphism of P so it has the form $f = c \circ f_\pi$, where c is the complementation operation on P and f_π is defined above. As above we compute $\operatorname{Tr}(f_0) = 1$ and $\operatorname{Tr}(f_{n-2}) = (-1)^{n-1} \operatorname{sgn}(\pi)$ so that $\Lambda(f) = 1 - \operatorname{sgn}(\pi)$. The fixed points of $f^2 = f_\pi^2 = f_{\pi^2}$ are unions of orbits of π^2. Now odd orbits of π give rise to odd orbits of π^2, while the even orbits of π give rise to a pair of orbits of π^2. There are three cases to consider for fixed points of f^2:

(1) $x \in P$ is an even orbit of π. Note that $\hat{1} \notin P$, so this does not include the case when all of π is an even cycle. Then x is the disjoint union of two orbits x_1, x_2 of π^2. Now $f(x_1) = (c \circ f_\pi)(x_1) = c(x_2) > x_1$. Thus if π has a proper even orbit, then there are elements $x \in P$ such that $x = f^2(x) < f(x)$.

(2) $x \in P$ is an odd orbit of π. Then $f(x) = (c \circ f_\pi)(x) = c(x)$ is not comparable with x. Moreover x contains no proper subset invariant under f^2.

(3) π is an even cycle. Then π^2 has two orbits x_1 and x_2. Both are invariant under f: $f(x_1) = (c \circ f_\pi)(x_1) = c(x_2) = x_1$.

Thus, when π is a product of odd cycles or when π is a single even cycle, Theorem 1.2 will give exact information about the fixed points of f. In the former case, $\operatorname{sgn}(\pi) = 1$, $\Lambda(f) = 0$ and f has no fixed points. In the latter case, $\operatorname{sgn}(\pi) = (-1)^{n-1} = -1$ (because n is even), $\Lambda(f) = 2$ and f has exactly two fixed points.

Finally suppose that $f: P \to P$ is order-reversing and not surjective. As before, one cannot characterize the form of f very easily; however, we again have $\Lambda(f) = 1$. Thus either there is some $x \in P$ for which $x = f^2(x) < f(x)$ or there is a *unique* fixed point of f. ▌

2. The Strong Fixed Point Property

A poset P has the *fixed point property* if every order-preserving map $f: P \to P$ has a fixed point. By convention, the empty set does not have the fixed point property. If P has the fixed point property and $g: P \to P$ is order-reversing then, since g^2 is order-preserving, g either satisfies $x = g^2(x) \neq g(x)$ for some $x \in P$ or g has a fixed point. Let us say that a finite poset P has the *strong fixed point property* if the following two conditions hold:

(i) every order-preserving map $f: P \to P$ satisfies $\mu(P^f) = 0$, in particular f has a fixed point,

(ii) every order-reversing map $g: P \to P$ satisfies $\mu(P_g) = 0$, in particular either $x = g^2(x) < g(x)$ for some $x \in P$ or g has a *unique* fixed point.

A poset P for which $H_i(P, K) = 0$ for $i \neq 0$ and $H_0(P, K) = K$ is said to be *acyclic over the field* K or K-*acyclic*. By the universal coefficient theorem, if P is K-acyclic for some field K, then P is $\mathbb{Q}$-acyclic. To get the following theorem in its strongest form it is therefore sufficient to consider the case $K = \mathbb{Q}$.

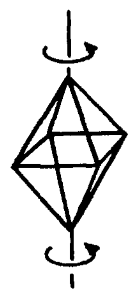
Theorem 2.1. *If the finite poset P is $\mathbb{Q}$-acyclic then P has the strong fixed point property.*

Proof. $\Lambda(f) = 1$ for any order-preserving or order-reversing map $f: P \to P$ since $f_0: \mathbb{Q} \to \mathbb{Q}$ has to be the identity and $f_i = 0$ for $i \neq 0$. The result is therefore directly implied by Theorems 1.1 and 1.2. ▌

If P is $\mathbb{Q}$-acyclic but infinite, then P need not have the fixed point property. Consider the set of integers under the usual order. This poset is clearly acyclic and even contractible but has a fixed point free automorphism.

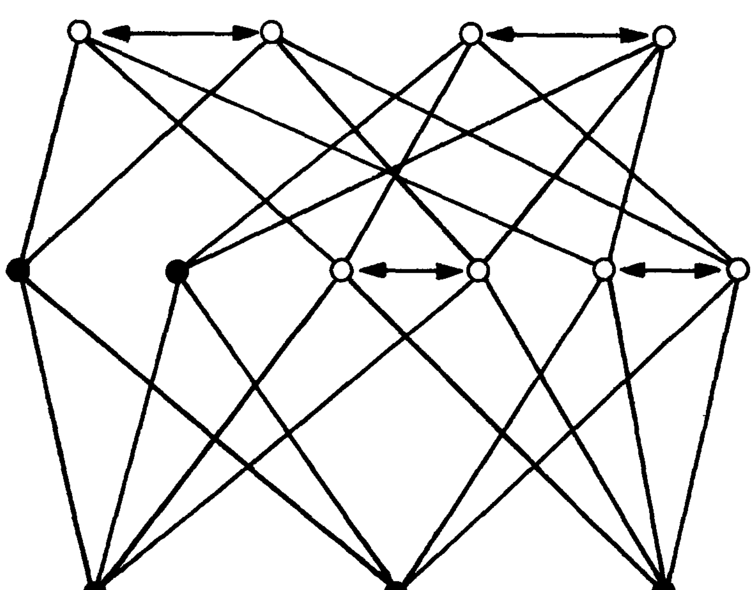
Theorem 2.1 implies that the fixed point set of every order-preserving self-map of a finite $\mathbb{Q}$-acyclic poset has Euler characteristic equal to 1. One naturally wonders whether the stronger condition of $\mathbb{Q}$-acyclicity is inherited by fixed point sets. The following example shows that this need not be the case.

EXAMPLE 2.2. Let $\hat{P}$ be the lattice of faces of an octahedron.



Let the automorphism $f: P \to P$ be given by rotating the octahedron 180° about an axis as shown. Let Q be obtained from P by identifying antipodal faces. Then the geometric realization $|Q|$ of Q is the real projective plane, and Q is therefore $\mathbb{Q}$-acyclic. Now f defines an automorphism $g: Q \to Q$, whose fixed point set Q^g has the property that $|Q^g|$ is the disjoint union of a circle and a point. Therefore Q^g is not $\mathbb{Q}$-acyclic.

In the following picture of Q, the action of g is indicated by arrows, and Q^g is marked with filled dots:



∎

Theorem 2.1 derives the fixed point property from an algebraic condition on the poset. In our next result we shall instead use a topological condition. A topological space X is said to have the fixed point property if every continuous map $f\colon X \to X$ has a fixed point.

THEOREM 2.3. *Let P be a poset. If $|P|$ has the fixed point property then so does P.*

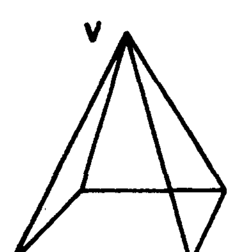
Proof. Assume that $|P|$ has the fixed point property and that $f\colon P \to P$ is order-preserving. Then the linear extension $|f|\colon |P| \to |P|$ has a fixed point $x \in |P|$ which belongs to a unique open simplex $\{p_0 < p_1 < \cdots < p_n\}$, $n \geqslant 0$, i.e., $x = \sum_{i=0}^{n} \lambda_i p_i$, $\sum_{i=0}^{n} \lambda_i = 1$ and $\lambda_i > 0$ for $i = 0, 1,..., n$. $\sum_{i=0}^{n} \lambda_i p_i = x = f(x) = \sum_{i=0}^{n} \lambda_i f(p_i)$ implies $\{p_i\}_{i=0}^{n} = \{f(p_i)\}_{i=0}^{n}$ and hence, since f is order-preserving, $p_i = f(p_i)$ for $i = 0, 1,..., n$. ∎

A poset for which the space $|P|$ has the homotopy type of a point is said to be *contractible*. Consider the following statements about a finite poset P.

A: P is contractible,
B: P is $\mathbb{Q}$-acyclic,
C: $|P|$ has the fixed point property,
D: P has the fixed point property.

Then $A \Rightarrow B \Rightarrow C \Rightarrow D$. Since P triangulates the compact polyhedron $|P|$ the first two implications are standard results in algebraic topology; the third is Theorem 2.3. Note that finiteness is needed only for the implication $B \Rightarrow C$. I. Rival [11, p. 310] has shown that all three implications are reversible when P has length equal to one. In general this is, of course, not true. For instance, let the poset P_r triangulate the real projective plane, and let the poset P_c triangulate the complex projective plane. Then P_r is $\mathbb{Q}$-acyclic but not contractible, while $|P_c|$ has the fixed point property without P_c being $\mathbb{Q}$-acyclic. In fact, $\chi(P_c) = 3$ which shows that if $|P|$ has the fixed point property and P is finite, it need not be the case that P has the strong fixed point property. Finally, the following example shows that D does not imply C when $\ell(P) \geqslant 2$.

EXAMPLE 2.4. Let $\hat{P}$ be the lattice of faces of the square pyramid.



If f is the identity map of P then $\mu(P^f) = \mu(P) = 1$. Let g be a bijection from the 0-faces (vertices) to the 2-faces of the pyramid such that u is not a face of $g(u)$ for all vertices u (two such bijections exist). Then g can be extended to an order-reversing bijection $g\colon P \to P$ such that $x = g^2(x) \leqslant g(x)$ is never satisfied, so $\mu(P_g) = -1$. Hence, P fails to satisfy both of the defining conditions for the strong fixed point property. Furthermore, $|P|$ does not have the fixed point property because $\Delta(P)$ triangulates S^2.

Nevertheless P has the fixed point property, as we now show. Let f be an order-preserving self-map of P. If f is not surjective then, as in Example 1.3, $\Lambda(f) = 1$ and f has at least one fixed point. On the other hand, if f is surjective then f is an automorphism. Therefore the vertex v must be a fixed point since v is covered by 4 elements in P and every other $x \in P$ is covered by at most 3 elements. ∎

For the rest of this section, and also in the beginning of the next section, we will describe a number of conditions on a finite poset which imply acyclicity and hence, by Theorem 2.1, the strong fixed point property. The fixed point theorems which arise include strengthened versions of some known results as well as new fixed point theorems in which no homological assumptions are explicitly made.

If one element x of a finite poset P is comparable to all other elements (e.g., if P is a lattice) it is easy to show by a direct combinatorial argument that P must have the fixed point property. In topology $\Delta(P)$ is called the *star* of x in this case. Since stars are contractible, P must in fact have the strong fixed point property. We will prove a number of far-reaching generalizations of this basic situation.

Let S be a subset of a poset P. An element x of P is an *upper* [*lower*] *bound* of S if $s \leqslant x$ [$s \geqslant x$] for all $s \in S$. We say S is *bounded* if there is either an upper or a lower bound of S in P. A subset C of P is said to be a *cutset* of P if every finite chain of P can be extended to a chain whose intersection with C is nonempty. A cutset C of P is *coherent* if every nonempty bounded subset T of C has either a join or a meet in P (note that it is not required that T have both a join and a meet in case T has both an upper and a lower bound in P). Finally, a subset A of P is said to be *astral* if there is some $x \in P$ such that x and a are comparable for all $a \in A$, i.e., if A is a subset of the star of x. If A is an antichain, i.e., a subset of P no two distinct elements of which are comparable, then A is astral if and only if A is bounded.

COROLLARY 2.5. *Let C be a coherent cutset in a finite poset P. Let $F_1, F_2, \ldots, F_n$ be the maximal astral subsets of C. Assume there is a nonempty astral subset S of C such that if $A = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $1 \leqslant k \leqslant n$, then either $A \cap S \neq \varnothing$ or $A \cup S$ is astral. Then P has the strong fixed point property.*

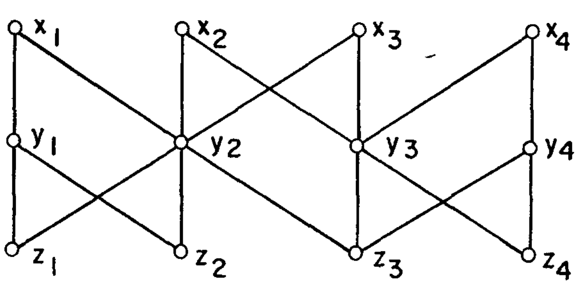
Proof. Under these assumptions it is proved in Björner [5, Theorem 3.2] that P is contractible. ▮

The condition in Corollary 2.5 is somewhat cumbersome. We will therefore consider a few simple cases. See also the proof of Proposition 3.1. When P has an element x which is comparable to all other elements of P then $S = C = \{x\}$ will fulfill the requirements. When the maximal astral subsets $F_1, F_2, \ldots, F_n$ of a coherent cutset C have nonempty intersection, then the condition is satisfied with $S = F_1 \cap F_2 \cap \cdots \cap F_n$. The special case when $F_1 = C$ (and hence $n = 1$) permits a particularly simple formulation:

COROLLARY 2.6. *Let C be a cutset in a finite poset P such that every nonempty subset of C has a join or a meet. Then P has the strong fixed point property.*

Corollary 2.6 generalizes and strengthens the finite version of a result of Höft and Höft [9, Corollary to Theorem 2], who considered the special case when C is the cutset of all minimal elements of P. The general case of their result is subsumed under Corollaries 4.3 and 5.3 below.

EXAMPLE 2.7. To illustrate the concepts we have introduced, consider the following poset:



The cutsets $\{x_1, x_2, x_3, x_4\}$ and $\{z_1, z_2, z_3, z_4\}$ are not coherent. The cutset $\{y_1, y_2, y_3, y_4\}$ is coherent, and its maximal astral subsets are $\{y_1, y_2\}$ and $\{y_2, y_3, y_4\}$. Hence, Corollary 2.5 applies. Every nonvoid subset of the cutset $\{x_1, y_2, y_3, x_4\}$ has a meet so Corollary 2.6 also applies. ▮

An element x of a finite poset P is said to be *irreducible* if x is covered by exactly one element or x covers exactly one element in P. Recall that x is said to *cover* y in P if $y < x$ and $y < z \leqslant x$ implies $z = x$. Let $I(P)$ be the set of irreducibles of P. A finite poset P is *dismantlable by irreducibles* if the elements of P can be ordered $a_1, a_2, \ldots, a_n$ in such a way that $a_1 \in I(P)$ and $a_i \in I(P - \{a_1, a_2, \ldots, a_{i-1}\})$ for $i = 2, 3, \ldots, n-1$. Examples of posets which are dismantable by irreducibles include connected posets P such that $\hat{P}$ is a planar lattice and also Example 2.7 above. I. Rival [11, Corollary 2] has proved that if P is dismantlable by irreducibles then P has the fixed point property.

COROLLARY 2.8. *If P is dismantlable by irreducibles then P has the strong fixed point property.*

Proof. Such a poset is contractible (Björner [5, Theorem 4.2]). ∎

The converse to Corollary 2.8 is not true, even when $\hat{P}$ is a lattice. For example, let P be the poset of faces of any triangulation of the real projective plane. Then P has the strong fixed point property by Theorem 2.1 and $\hat{P}$ is a lattice, but P has no irreducibles.

A finite poset P is called an *n-bouquet of spheres over* $\mathbb{Q}$ if the following hold:

(1) if $n = -1$, then $P = \varnothing$;
(2) if $n = 0$, then P is a nonempty antichain;
(3) if $n > 0$, then P is connected and $H_i(P, \mathbb{Q}) = 0$ for $i \neq 0, n$.

A finite poset P is *Cohen–Macaulay over* $\mathbb{Q}$ if every open interval $(x, y) = \{z \in P \mid x < z < y\}$ of $\hat{P}$ is an $\ell(x, y)$-bouquet of spheres over $\mathbb{Q}$, where $\ell(x, y)$ is the length of the open interval (x, y), which we take to be -1 if (x, y) is empty (cf. Baclawski [3]). Known examples of Cohen–Macaulay posets include finite posets P such that $\hat{P}$ is a semimodular lattice, the face-lattice of a convex polytope or the lattice of subgroups of a supersolvable group.

COROLLARY 2.9. *If P is Cohen–Macaulay over $\mathbb{Q}$ and $\mu(P) = 0$, then P has the strong fixed point property.*

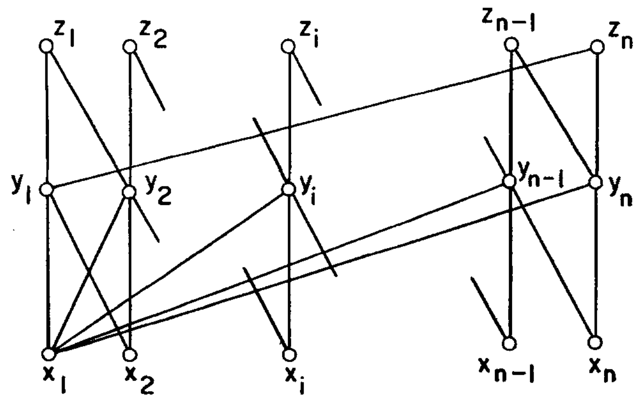
Proof.

$$0 = \mu(P) = \chi(P) - 1 = -1 + \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(P, \mathbb{Q})$$

$$= \begin{cases} -1 + \dim_{\mathbb{Q}} H_0(P, \mathbb{Q}), & \text{if } \ell(P) = 0 \\ (-1)^\ell \dim_{\mathbb{Q}} H_\ell(P, \mathbb{Q}), & \text{if } \ell(P) = \ell > 0. \end{cases}$$

Hence in either case P is $\mathbb{Q}$-acyclic. ∎

EXAMPLE 2.10. Define posets P_n for $n \geqslant 3$ by the following diagram:



These posets were considered by Duffus, Poguntke and Rival [8, Fig. 4]. The special case P_3 also appears in Rival [11, Fig. 2]. There are at least two quick ways to see that P_n has the strong fixed point property. Observe that every nonempty subset of the cutset $\{y_1, y_2, \ldots, y_n\}$ has a meet, except for the subsets $\{y_i, y_{i+1}\}$, $i = 1, 2, \ldots, n-1$, which have a join. Hence, Corollary 2.6 applies. An alternative is to use Corollary 2.9. A direct computation yields that $\mu(P_n) = 0$ and the homotopy theorem for Cohen–Macaulay posets (see Baclawski [3]) makes it easy to see that P_n is Cohen–Macaulay. ∎

3. Fixed Points and Complements in Lattices

Let L be a lattice with $\hat{0}$ and $\hat{1}$. For $x \in L$ we say that $y \in L$ is a *complement* of x if $x \vee y = \hat{1}$ and $x \wedge y = \hat{0}$. The lattice L is *complemented* if every $x \in L$ has a complement (see Birkhoff [4, p. 16]) and is *noncomplemented* otherwise. It is known that if $\hat{P}$ is a finite noncomplemented lattice, then P is acyclic (Baclawski [2, Corollary 6.3]) and in fact is even contractible (Björner [5, Theorem 3.3]). Such a poset P therefore has the strong fixed point property by Theorem 2.1. We will examine some generalizations and some consequences of this basic result in this section. We begin with two generalizations.

Let M be the cutset of all minimal elements of a finite poset P. Assume that M is coherent, and let P_M be the subposet of P consisting of all elements that can be obtained as joins of nonempty subsets of M. It is not hard to see that $\hat{P}_M$ is a lattice.

Proposition 3.1. *Let P be a finite poset. Assume that the cutset M of minimal elements is coherent. If $\hat{P}_M$ is noncomplemented, then P has the strong fixed point property.*

Proof. Let $z \in P_M$ be an element lacking a complement in $\hat{P}_M$. Let $S = \{x \in M \mid x \leqslant z\}$. Then S is bounded (hence astral). We wish to apply Corollary 2.5. To do so we must examine the maximal astral subsets of M. Clearly these have the form $\{x \in M \mid x \leqslant m\}$ for some maximal element m of P_M. Let $A = \bigcap_{i=1}^{k} \{x \in M \mid x \leqslant m_i\}$, where m_i is a maximal element of P_M for all i. Then $A = \{x \in M \mid x \leqslant a\}$, where $a = m_1 \wedge \cdots \wedge m_k$ in $\hat{P}_M$. Now by the choice of z, either $a \wedge z > \hat{0}$, in which case $A \cap S \neq \varnothing$, or $a \vee z < \hat{1}$, in which case $A \cup S$ is bounded. The hypotheses of Corollary 2.5 are therefore satisfied. ∎

Recall that an element of a lattice is an *atom* if it covers $\hat{0}$ and a *coatom* if it is covered by $\hat{1}$. We will say that a finite lattice is *strongly complemented* if every element has a complement which is a join of atoms as well as one which is a meet of coatoms. Of course, here we consider $\hat{0}$ to be the join of the empty set of atoms and dually for $\hat{1}$.

COROLLARY 3.2. *Let $\hat{P}$ be a finite lattice which is not strongly complemented. Then P has the strong fixed point property.*

Proof. Without loss of generality, we may assume that there is an element y of P that has no complement which is a join of atoms. The cutset M of P consisting of all minimal elements is coherent because $\hat{P}$ is a lattice. If $\hat{P}_M$ were complemented, then any complement of $z = \bigvee \{x \in M \mid x \leqslant y\}$ in $\hat{P}_M$ would also be a complement of y in $\hat{P}$. But every element of P_M is a join of atoms so this contradicts the choice of y. Hence $\hat{P}_M$ is not complemented, and we may therefore apply Proposition 3.1. ∎

Let P be a poset. A self-map f: $P \to P$ is said to be a *friendship map* if

(1) f is order-reversing, and

(2) x and $f(x)$ are incomparable for all $x \in P$, except for $\hat{0}$ and/or $\hat{1}$ if they exist.

We will say that a poset P is *friendly* if it possesses a friendship map.

THEOREM 3.3. *Every finite friendly lattice is strongly complemented.*

Proof. Let $L = \hat{P}$ be the lattice and let $f: L \to L$ be its friendship map. We first note that by condition (2) of the definition of a friendship map, f restricts to a friendship map f: $P \to P$. Now f is order-reversing and cannot satisfy $x = f^2(x) \leqslant f(x)$ for any $x \in P$. Therefore P does not have the strong fixed point property. By Corollary 3.2, L is strongly complemented. ∎

In the proof of Theorem 3.3 we used only that f restricts to $P = L - \{\hat{0}. \hat{1}\}$ and that x and $f(x)$ are incomparable for x in P^{f^2}. In Example 1.3 we saw that there are numerous maps of this kind on a finite Boolean algebra. Non-Boolean finite distributive lattices do not admit any friendship maps at all, even in this weaker sense. However, the class of geometric lattices is more amicable in this respect, as we now show.

THEOREM 3.4. *Let L be a lattice of finite length. Suppose that there is a set $A = \{a_1, a_2, \ldots, a_n\}$ of atoms of L such that*

(1) *for all i, $a_1 \vee \cdots \vee \hat{a}_i \vee \cdots \vee a_n \neq \hat{1}$, where $\hat{a}_i$ means "omit a_i,"*

(2) *$\hat{0}, a_1, a_1 \vee a_2, \ldots, a_1 \vee a_2 \vee \cdots \vee a_n = \hat{1}$ is a maximal chain of L.*

Then L is friendly.

Proof. We may assume that $n > 1$. Define a function g from the set C of coatoms of L to A by: $g(c) = a_i$, where i is the smallest integer such that $c \not\geqslant a_i$. Note that g is well-defined, for if c were above all the atoms of A, this would imply that $c \geqslant \hat{1}$, by condition (2). Note also that $c \not\geqslant g(c)$ for all $c \in C$.

Let $L = \hat{P}$. Extend g to a map $f\colon P \to L$ by

$$f(x) = \bigvee \{g(c) \mid c \geqslant x,\ c \in C\}.$$

We will show that $f\colon P \to L$ satisfies the conditions:

(a) f is order-reversing,
(b) $f(P) \subseteq P$,
(c) $x \ngeqslant f(x)$, for all $x \in P$,
(d) $x \nleqslant f(x)$, for all $x \in P$.

Now (a) is obvious. To show (c) suppose that $x \geqslant f(x)$. Let d be a coatom such that $d \geqslant x$, then

$$d \geqslant x \geqslant f(x) = \bigvee \{g(c) \mid c \geqslant x,\ c \in C\}.$$

In particular, $d \geqslant g(d)$. This contradicts the choice of g. Thus (c) holds. As a special case, $\hat{0} \notin f(P)$.

To show (b) we need only show that $\hat{1} \notin f(P)$. Suppose that $x \in P$ satisfies $f(x) = \hat{1}$. By hypothesis (1), this implies that $\{g(c) \mid c \geqslant x\} = A$. For each i, $1 \leqslant i \leqslant n$, choose an element c_i of C so that $g(c_i) = a_i$ and $c_i \geqslant x$. First consider c_n. By definition of g, $c_n \ngeqslant a_n$ and $c_n \geqslant a_1, \ldots, a_{n-1}$. Hence $c_n \geqslant a_1 \vee \cdots \vee a_{n-1}$. By hypothesis (2), $a_1 \vee \cdots \vee a_{n-1}$ is a coatom, so $c_n = a_1 \vee \cdots \vee a_{n-1}$. Hence $x \leqslant a_1 \vee \cdots \vee a_{n-1}$.

Assume that $x \leqslant a_1 \vee \cdots \vee a_i$, where $i < n$. By definition of g, $c_i \ngeqslant a_i$, but $c_i \geqslant a_1, \ldots, a_{i-1}$ so $c_i \geqslant a_1 \vee \cdots \vee a_{i-1}$ and $c_i \ngeqslant a_1 \vee \cdots \vee a_i$. Hence $c_i \wedge (a_1 \vee \cdots \vee a_i) < a_1 \vee \cdots \vee a_i$, while $a_1 \vee \cdots \vee a_{i-1} \leqslant c_i \wedge (a_1 \vee \cdots \vee a_i)$. By hypothesis (2), we conclude that $a_1 \vee \cdots \vee a_{i-1} = c_i \wedge (a_1 \vee \cdots \vee a_i)$. Now $x \leqslant a_1 \vee \cdots \vee a_i$ by assumption and $x \leqslant c_i$ by choice of c_i. Thus $x \leqslant c_i \wedge (a_1 \vee \cdots \vee a_i) = a_1 \vee \cdots \vee a_{i-1}$. By descending induction, $x \leqslant a_1$. But $x \leqslant c_1$ and $c_1 \ngeqslant a_1$. Hence $x \leqslant a_1 \wedge c_1 = \hat{0}$. This contradicts the fact that $x \in P$.

It remains to show (d). Let $x \in P$ satisfy $x \leqslant f(x)$. Let d be a coatom such that $d \geqslant f(x)$. There is such a coatom by (b). Then

$$d \geqslant f(x) = \bigvee \{g(c) \mid c \geqslant x,\ c \in C\} \geqslant x.$$

But the above inequality implies that $d \geqslant g(d)$. This contradicts the choice of g. Thus we see that (d) holds and that f is a friendship map on P. The obvious extension of f to $L = \hat{P}$ is then a friendship map on L. ▌

COROLLARY 3.5. *Every geometric lattice is friendly.*

Proof. For the set A of atoms required in Theorem 3.4, any basis of the geometry will do. ▌

We end this section with an example of a connection between lattice complementation and fixed point theory which is somewhat different from what we have considered so far. The following result is the finite version of a unique fixed point theorem due to Wong [15, Theorem 5].

PROPOSITION 3.6 (Wong). *Let L be a finite lattice such that every lower interval $[\hat{0}, x]$, $x \in L$, is complemented. Let $f: L \to L$ be a mapping such that*

(i) $f(x \vee y) = f(x) \wedge f(y)$ *for all* $x, y \in L$,
(ii) $f(x) \wedge x \neq \hat{0}$ *when* $x \neq \hat{0}$.

Then f has a unique fixed point.

Proof. L is contractible and therefore has the strong fixed point property by Theorem 2.1. Condition (i) implies that f is order-reversing. Hence, all we need to show is that $x = f^2(x) < f(x)$ cannot occur. The following argument is due to Wong. Assume that $x = f^2(x) < f(x)$. Let z be a complement of x in the interval $[\hat{0}, f(x)]$. Then $z \neq \hat{0}$ and $z \wedge f(z) = (z \wedge f(x)) \wedge f(z) = z \wedge (f(x) \wedge f(z)) = z \wedge f(x \vee z) = z \wedge f^2(x) = z \wedge x = \hat{0}$, which contradicts (ii). ∎

4. Dismantlable Posets

In Section 2 we observed that infinite contractible posets need not have the fixed point property. On the other hand, most known examples of infinite posets having the fixed point property are contractible. It seems natural to ask what conditions in addition to contractibility are needed to ensure that an infinite poset have the fixed point property. In this section we take a first step in this direction by introducing a class of contractible posets which contains the complete lattices and which turns out to be well-behaved with respect to the fixed point property.

The motivation for this section arose from our work on products in Section 5 and from a desire to improve an earlier version of Corollary 4.3. Theorem 4.1 was found jointly by us and Rival [12], and this led to our use of the term "dismantlable." In the finite case, all the results of this section, except for Corollary 4.3 and part of Theorem 4.1, are due to Rival [11], [12] and to Duffus and Rival [16].

For any two posets P and Q, we write P^Q for the poset of order-preserving maps $f: Q \to P$, partially ordered by $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in Q$. In Birkhoff [4, p. 55] the poset P^Q is called the *cardinal power* with base P and exponent Q. Later in this section we consider the question of when P^Q has the fixed point property.

Recall a standard notion from set theory: a chain is [*dually*] *well-ordered* if every nonempty subset of it has a minimum [maximum] element. We will say that a poset P is *well-ordered-complete* if every nonvoid well-ordered chain of P has a join in P and every nonvoid dually well-ordered chain of P has a meet in P.

If a and b are elements of a poset P, we say that a is *connected to* b if there exist elements $c_1, c_2, \ldots, c_n$ of P such that $a \leqslant c_1 \geqslant c_2 \leqslant \cdots \geqslant c_n \leqslant b$. A poset P is said to be *connected* if all elements of P are pairwise connected to each other. The following convention will be helpful in this section: if P and Q are two posets and $p \in P$ we agree that $\bar{p} \in P^Q$ denotes the constant map which sends all $q \in Q$ to p.

We will say that a poset P is *dismantlable* if

(1) P is well-ordered-complete, and

(2) the identity map of P is connected to some constant map in P^P.

The second condition above may be rephrased as requiring that there exist order-preserving self-maps $f_1, f_2, \ldots, f_n$ of P and an element p of P such that

$$x \leqslant f_1(x) \geqslant f_2(x) \leqslant f_3(x) \geqslant \cdots \leqslant f_n(x) \geqslant p,$$

for all $x \in P$. Using Quillen's "homotopy property" [10, 1.3] it is immediate from condition (2) that all dismantlable posets are contractible. Hence, by Theorem 2.1 all *finite* dismantlable posets have the strong fixed point property. This observation is equivalent to Corollary 2.8 by the following result, which shows that dismantlability extends to arbitrary posets the notion of "dismantlability by irreducibles" defined for finite posets in Section 2.

THEOREM 4.1. *A finite poset is dismantlable if and only if it is dismantlable by irreducibles.*

Proof. Assume that P is dismantlable by irreducibles. Let $a_1, a_2, \ldots, a_k$ be an ordering of the elements of P such that a_i is an irreducible in $P - \{a_1, a_2, \ldots, a_{i-1}\}$ for $i = 1, 2, \ldots, k-1$. Define maps $g_i \colon P - \{a_1, a_2, \ldots, a_{i-1}\} \to P - \{a_1, a_2, \ldots, a_i\}$ for $i = 1, 2, \ldots, k-1$ in the following way: if a_i covers [is covered by] exactly one element b_i in $P - \{a_1, a_2, \ldots, a_{i-1}\}$ let

$$\begin{aligned} g_i(x) &= x, \qquad \text{if } x \neq a_i \\ &= b_i, \qquad \text{if } x = a_i. \end{aligned}$$

Then g_i is order-preserving and $g_i(x) \leqslant x [g_i(x) \geqslant x]$ for all $x \in P - \{a_1, a_2, \ldots, a_{i-1}\}$. Let $h_i \colon P - \{a_1, a_2, \ldots, a_i\} \to P$ be the inclusion map. Now define self-maps f_i of P for $i = 0, 1, \ldots, k-1$ by letting f_0 be the identity map of P and $f_i = h_i \circ g_i \circ g_{i-1} \circ \cdots \circ g_1$ for $i > 0$. Then the maps f_i are order-preserving, and either $f_i(x) \leqslant f_{i+1}(x)$ for all $x \in P$ or $f_i(x) \geqslant f_{i+1}(x)$ for all $x \in P$. Hence the maps f_i connect the identity map f_0 of P to the constant map $f_{k-1} = \bar{a}_k$ in P^P.

Now assume that there exist $f_1, f_2, \ldots, f_n \in P^P$ which connect the identity map of P to the constant map $\bar{p}$, for some $p \in P$. We may, without loss of generality, assume that $x < f_1(x)$ for some $x \in P$. Let a be a maximal element of $\{x \mid x < f_1(x)\}$. If we define $g\colon P \to P$ by

$$\begin{aligned} g(x) &= x, && \text{if } x \neq a \\ &= f_1(a), && \text{if } x = a, \end{aligned}$$

then it is easy to see that $g \in P^P$ and that $g(a)$ covers a. Suppose that a were covered by another element b. Now $b \geqslant a$ implies that $g(b) \geqslant g(a)$, but $g(b) = b$ by definition and two distinct elements which both cover a cannot be comparable. Hence, a is covered by exactly one element and is therefore irreducible.

Let $P' = P - \{a\}$. Define self-maps f_i' of P' for $i = 1, 2, \ldots, n$ by

$$\begin{aligned} f_i'(x) &= f_i(x), && \text{if } f_i(x) \neq a \\ &= g(a), && \text{if } f_i(x) = a. \end{aligned}$$

The maps f_i' are order-preserving and connect the identity map of P' to $\overline{g(p)}$. By induction on the number of elements of P, the poset P is dismantlable by irreducibles. ∎

If some element of a well-ordered-complete poset P is comparable to all other elements then P is dismantlable. More generally, if some element p of a well-ordered-complete poset P has a join $p \vee x$ with every other element x of P then P is dismantlable (take $f_1(x) = p \vee x$). For example, every complete lattice is dismantlable.

THEOREM 4.2. *Every dismantlable poset has the fixed point property.*

Proof. We will need to refer to the following technical result: if an order-preserving self-map f of a well-ordered-complete poset Q satisfies $a \leqslant f(a)$ or $a \geqslant f(a)$ for some $a \in Q$ then f has a fixed point. A brief proof based on Zorn's lemma may be found in Wong [15, Theorem 1]. A more involved proof that does not use any form of the Axiom of Choice is given by Abian and Brown [1, Theorem 2].

Assume that P is a dismantlable poset and that f is an order-preserving self-map of P. There are, by definition, order-preserving self-maps g_i, $i = 0, 1, \ldots, n$ of P which connect the identity map g_0 of P to a constant map $g_n = \bar{p}, p \in P$, in P^P. Define order-preserving self-maps h_i of P for $i = 0, 1, \ldots, n$ by $h_i = g_i \circ f$. The map h_n has the fixed point p. Now assume that h_{k+1} has a fixed point z. Then $h_k(z) \leqslant h_{k+1}(z) = z$ or $h_k(z) \geqslant h_{k+1}(z) = z$. Hence, since P is well-ordered-complete, h_k has a fixed point. By finite induction, therefore, $h_0 = f$ has a fixed point. ∎

As an application of Theorem 4.2, we can extend part of Corollary 2.6 to the infinite case.

COROLLARY 4.3. *Let P be a well-ordered-complete poset. Assume that there exists a cutset C of P such that*

(a) *C is an antichain,*
(b) *every nonempty subset of C has a join in P.*

Then P has the fixed point property.

Proof. It suffices to show that such a poset P is dismantlable. Let D be the order-filter generated by the cutset C in P, i.e., $D = \{x \in P \mid x \geqslant c$ for some $c \in C\}$. Define a self-map f of P by

$$\begin{aligned} f(x) &= x, && \text{if } x \notin D \\ &= \bigvee \{c \in C \mid c \leqslant x\}, && \text{if } x \in D. \end{aligned}$$

If $p = \vee C$ then for every $x \in P$, $x \geqslant f(x) \leqslant p$. Hence, it remains to show only that f is order-preserving.

Assume that $x \leqslant y$ in P. We must consider three cases.

(1) $y \notin D$. Then $x \notin D$, and $f(x) = x \leqslant y = f(y)$.

(2) $y \in D$ and $x \notin D$. Since C is a cutset, the two-element chain $\{x, y\}$ can be extended to a chain which includes an element, say d, of C. $d \leqslant x$ cannot occur since $x \notin D$. $y < d$ cannot occur since, by assumption, $y \geqslant c$ for some $c \in C$ and $c \leqslant y < d$ contradicts C being an antichain. Hence, $x < d \leqslant y$ and therefore $f(x) = x < d \leqslant f(y)$.

(3) $y \in D$ and $x \in D$. $x \leqslant y$ implies that $\{c \in C \mid c \leqslant x\} \subseteq \{c \in C \mid c \leqslant y\}$ and hence that $f(x) \leqslant f(y)$.

So f is order-preserving, and P is dismantlable. ∎

We end this section by giving a variety of equivalent formulations of the notion of dismantlability in terms of properties of cardinal powers of posets. We begin with a lemma which allows us to deduce fixed point properties of cardinal powers.

LEMMA 4.4. *If P is dismantlable and Q is a nonempty poset, then P^Q is dismantlable.*

Proof. Assume that P is dismantlable and let Q be a nonvoid poset. Let us first verify that P^Q is well-ordered-complete. Let $\mathscr{C}$ be a nonvoid well-ordered chain in P^Q. Then for each $x \in Q$, $\{h(x) \mid h \in \mathscr{C}\}$ forms a well-ordered chain in P and so, by assumption, has a join in P. We may therefore define a map $g\colon Q \to P$ by $g(x) = \vee \{h(x) \mid h \in \mathscr{C}\}$. It is easy to check that g is order-preserving and that g is the join of $\mathscr{C}$ in P^Q. A dual argument shows that every nonvoid dually well-ordered chain in P^Q has a meet.

Let f_i, $i = 0, 1,..., n$, be order-preserving self-maps of P which connect the identity map f_0 of P to a constant map $f_n = \bar{p}$, $p \in P$, in P^P. Define self-maps F_i, $i = 0, 1,..., n$, of P^Q by $F_i(g) = f_i \circ g$. It is straightforward to verify that the maps F_i are order-preserving and that they connect the identity map F_0 of P^Q to the constant map $F_n = \overline{(\bar{p})}$ in $(P^Q)^{P^Q}$. Hence, P^Q is dismantlable. ∎

THEOREM 4.5. *Let P be a well-ordered-complete poset. Then the following conditions are equivalent*:

(i) *P is dismantlable,*
(ii) *P^Q has the fixed point property for every nonempty poset Q,*
(iii) *P^Q is connected for every nonempty poset Q,*
(iv) *P^P has the fixed point property,*
(v) *P^P is connected.*

Proof. By combining Theorem 4.2 and Lemma 4.4 we get that (i) implies (ii). (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are trivial, since any poset with the fixed point property must be connected. If P^P is connected then, in particular, the identity map of P is connected to every constant map. Since P is by assumption well-ordered-complete we see that (v) implies (i). ∎

It is a consequence of this theorem that P^Q does not necessarily have the fixed point property, even if both P and Q do. Let P be any finite poset which is not dismantlable but has the fixed point property (several examples of such posets were given in Section 2). Then P^P does not have the fixed point property.

5. CONSTRUCTING POSETS HAVING THE FIXED POINT PROPERTY

In this section we will be concerned with two particular instances of the following general question: if a poset R is "constructed" from parts all of which have the fixed point property, does it follow that R does also? For example, we saw at the end of Section 4 that the answer is negative for the cardinal power construction P^Q, even when $P = Q$. The two constructions we will consider below are the "gluing together" of subposets via a fiber map and the direct product.

Let P and L be posets, and let $F: P \to L$ be an order-preserving map. We think of F as a way of "piecing together" or "constructing" P from the subposets

$$F/y = \{x \in P \mid F(x) \geqslant y\},$$

called the *fibers* of F over elements y in the *base* L. One can always piece together acyclic posets to get new ones.

THEOREM 5.1. *Let $F: P \to L$ be an order-preserving map of posets such that*

(a) *L is $\mathbb{Q}$-acyclic, and*

(b) *F/y is $\mathbb{Q}$-acyclic for all $y \in L$.*

Then P is $\mathbb{Q}$-acyclic.

Proof. See Baclawski [3]. ∎

The above theorem remains true if the word "$\mathbb{Q}$-acyclic" is replaced everywhere by "contractible." For a proof see Quillen [10, Proposition 1.6]. Note that if P is finite then, by Theorem 2.1, the assumptions made in Theorem 5.1 imply that P has the strong fixed point property.

We will next derive a similar theorem for the general fixed point property by requiring the fixed point property, rather than acyclicity, in L and the fibers of F. Unfortunately, to obtain such a theorem we are forced to impose strong restrictions on F and L. Moreover, Example 5.4 below shows that these restrictions cannot easily be relaxed.

THEOREM 5.2. *Let $F: P \to L$ be an order-preserving map from a poset P to an complete lattice L. Write $M \subseteq P$ for the minimal elements of P and L_M for the upper (join) sub-semilattice of L generated by $F(M)$. Assume that*

(a) *L_M has the fixed point property,*

(b) *for every $y \in L_M$, F/y has the fixed point property,*

(c) *for every $x \in P$ there is an $m \in M$ such that $m \leqslant x$,*

(d) *for every $m \in M$, the fiber over $F(m)$ is $\{x \in P \mid x \geqslant m\}$.*

Then P has the fixed point property.

Proof. Let $f: P \to P$ be an order-preserving self-map. We define a map $G: L_M \to L_M$ by

$$G(y) = \bigvee \{F(n) \mid n \in M \text{ and there exists } m \in M \text{ such that}$$

$$n \leqslant f(m) \text{ and } F(m) \leqslant y\}.$$

G is well-defined because of property (c). It is easy to see that G is order-preserving. By (a), G has a fixed point $y \in L_M$. We show that $f(F/y) \subseteq F/y$. Let $b \in F/y$ so that $F(b) \geqslant y$. The fact that $G(y) = y$ means that if

$$S = \{n \in M \mid n \leqslant f(m) \text{ for some } m \in M \text{ such that } F(m) \leqslant y\},$$

then $y = \bigvee F(S)$. Let $n \in S$. Choose $m \in M$ such that $n \leqslant f(m)$ and $F(m) \leqslant y$.

Then $F(b) \geqslant y \geqslant F(m)$, i.e., b is in the fiber of F over $F(m)$. By (d) therefore, $b \geqslant m$. Since f is order-preserving, $f(b) \geqslant f(m) \geqslant n$. Since F is order-preserving, $F(f(b)) \geqslant F(n)$. Since n was an arbitrary element of S, $F(f(b)) \geqslant \bigvee F(S) = y$. Hence $f(b) \in F/y$. Therefore $f(F/y) \subseteq F/y$ as claimed above. By (b), f has a fixed point in F/y. The result then follows. ∎

COROLLARY 5.3. *Let P be a poset with finitely many minimal elements M. Assume that*

(i) *for every non-empty subset S of M the subposet $\{x \in P \mid x \geqslant s$ for all $s \in S\}$ has the fixed point property,*

(ii) *for every $x \in P$ there is an $m \in M$ such that $m \leqslant x$.*

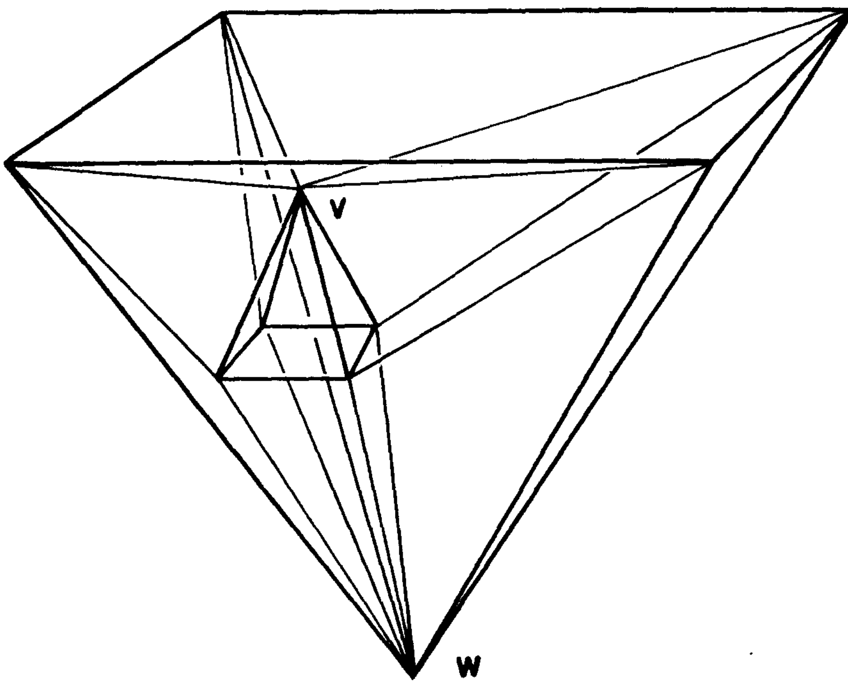
Then P has the fixed point property.

Proof. Let L be the Boolean algebra of all subsets of M. Define a map $F: P \to L$ by $F(x) = \{m \in M \mid m \leqslant x\}$. F is then order-preserving and $L_M = L - \{\hat{0}\}$. We want to check that conditions (a)–(d) of Theorem 5.2 are satisfied. (d) is clear from the construction of F and implies that (b) is equivalent to (i). (c) is the same as (ii). (a) also holds, since L_M is finite and has a maximum element. ∎

Duffus, Poguntke and Rival [8, Theorem 2] proved Corollary 5.3 in the special case when P is finite. We remark that conditions (a), (b) and (c) in Theorem 5.2 obviously cannot be eliminated. Whether the assumptions about L can be relaxed is open, but one generally uses some well-understood lattice to do such a construction. This leaves condition (d). Condition (d) is very strong. It implies that F is injective on M and that $F(M)$ is an antichain of L. Furthermore, in the finite case because of condition (d), Theorem 5.2 in general follows from Corollary 5.3, i.e., from the theorem of Duffus, Poguntke and Rival [8] cited above. However, their proof method does not extend to the infinite case. Furthermore our proof is constructive in the sense that if algorithms are known for finding a fixed point in L_M and in the fibers of F, then we give an algorithm for finding one in P.

In the following example we show that condition (d) cannot be eliminated even when L is B_2. This example also provides a counterexample to a conjecture of L. Mohler. Mohler's conjecture essentially states that Theorem 5.2 without condition (d) holds in case P is finite and $L = B_2$. For the exact formulation we refer to Duffus, Poguntke and Rival [8, p. 4].

EXAMPLE 5.4. Let C_1 and C_2 be two copies of a cone over the boundary of the square pyramid. We join these two together along their boundaries as indicated below:



In this figure two of the vertices are not shown. One vertex should be in the interior of the inside (right-side up) pyramid, joined to all of the vertices of that pyramid. This represents the cone C_1. Another vertex should be the point at infinity of 3-space, joined to all of the vertices of the outside (upside down) pyramid. This represents the cone C_2. The resulting figure is a regular cell decomposition of the 3-sphere S^3. We let $\hat{P}$ be the lattice of faces of this decomposition, so that $|\Delta(P)|$ is homeomorphic to S^3.

We can also construct $\hat{P}$ as the face-lattice of a convex polytope as follows. Begin with a cube in 3-space centered at the origin. Embed 3-space in 4-space by the map $(x, y, z) \mapsto (x, y, z, 0)$. Let v and w be the points $(0, 0, 0, \pm 1)$. Form the convex hull H of the cube and the points v and w. The vertices v and w are indicated in the figure above. Finally choose two antipodal 3-faces and two points outside H but sufficiently near the centers of these two 3-faces. Take the convex hull of H and these two new points. $\hat{P}$ is now the lattice of faces of this polytope.

We first observe that P does not have the fixed point property: the antipodal map is fixed point free. Next map P to the Boolean algebra B_2 by mapping the point at infinity to one atom and the remaining vertices to the other atom. Map each remaining element of P in an order-preserving way to an atom if it can be so mapped and to $\hat{1}$ if not. The fibers over the atoms of B_2 are contractible and so have the fixed point property. The fiber over $\hat{1}$ is isomorphic to the poset in Example 2.4. Hence it also has the fixed point property. Thus all the conditions of Theorem 5.2 are satisfied except for condition (d), but P does not have the fixed point property. ∎

A very interesting construction from the point of view of fixed point theory is the direct product (for the definition see Birkhoff [4, p. 8]). I. Rival has asked [12] whether the direct product preserves the fixed point property. The following results show that this is often true in familiar cases, but the general question remains open.

COROLLARY 5.5. *Let P and Q be $\mathbb{Q}$-acyclic posets. Then $P \times Q$ is $\mathbb{Q}$-acyclic.*

Proof. This is a straightforward application of the Künneth formula. Alternatively the result can be deduced from Theorem 5.1 as follows. Let $F\colon P \times Q \to P$ be the projection onto the first factor, $F(p, q) = p$. For any $p \in P$ the fiber F/p is the poset $V_p \times Q$, where $V_p = \{x \in P \mid x \geqslant p\}$. Now let $G\colon V_p \times Q \to Q$ be the projection onto Q. For any $q \in Q$ the fiber G/q is the poset $V_p \times V_q$, where $V_q = \{x \in Q \mid x \geqslant q\}$. $V_p \times V_q$ has the minimum element (p, q) and is therefore $\mathbb{Q}$-acyclic. By repeated application of Theorem 5.1, the result follows. ∎

THEOREM 5.6. *Let P and Q be posets such that Q and P^Q both have the fixed point property. Then $P \times Q$ also has the fixed point property.*

Proof. Let $f\colon P \times Q \to P \times Q$ be an order-preserving map. Let $F\colon P \times Q \to P$ and $G\colon P \times Q \to Q$ be the natural projections, as in the preceding proof. We define a map $\alpha\colon P^Q \to P^Q$ by

$$\alpha(g)(x) = F(f(g(x), x)),$$

for $g \in P^Q$ and $x \in Q$. Clearly α is well-defined. Let $g \leqslant h$ hold in P^Q. Then for all $x \in Q$,

$$\begin{aligned} g(x) &\leqslant h(x) \\ &\Rightarrow f(g(x), x) \leqslant f(h(x), x) \\ &\Rightarrow F(f(g(x), x)) \leqslant F(f(h(x), x)) \\ &\Rightarrow \alpha(g)(x) \leqslant \alpha(h)(x), \end{aligned}$$

i.e., $\alpha(g) \leqslant \alpha(h)$ in P^Q. Thus α is order-preserving. By the hypothesis on P^Q, α has a fixed point g_0.

Now we define a map $\beta\colon Q \to Q$ by

$$\beta(x) = G(f(g_0(x), x)),$$

for all $x \in Q$. Clearly β is order-preserving. By the hypothesis on Q, β has a fixed point x_0. Now

$$\begin{aligned} f(g_0(x_0), x_0) &= (F \circ f(g_0(x_0), x_0), G \circ f(g_0(x_0), x_0)) \\ &= (\alpha(g_0)(x_0), \beta(x_0)) \\ &= (g_0(x_0), x_0). \end{aligned}$$

Therefore $(g_0(x_0), x_0)$ is a fixed point of f. ∎

COROLLARY 5.7. *Let P be a dismantlable poset and Q a poset having the fixed point property. Then $P \times Q$ has the fixed point property.*

Proof. If P is dismantlable then P^Q has the fixed point property by Theorem 4.5. Hence, the above theorem applies. ∎

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