The Möbius Algebra as a Grothendieck Ring

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1. INTRODUCTION

The notion of the Möbius algebra as a setting for Möbius inversion is now a well-developed tool in Combinatorial theory. It was first defined by Solomon [16] who used it in the study of the Burnside algebra of a finite group. It was subsequently studied by Davis [8] and Greene [11]. Davis showed that the Möbius algebra and Rota's valuation ring [15] are equivalent, at least in the finite case. Greene showed that Rota's theorem [14] relating the Möbius functions of finite ordered sets joined by a Galois connection fits very naturally into this context. See Geissinger [10] for the infinite case. For numerous applications see Greene [11], Crapo-Roulet [7] and Zaslavsky [18].

Grothendieck groups and rings are well-known both in topology and in commutative algebra. They occur in the very general contexts of exact categories and of topoi. We only consider the relatively special case of the Grothendieck ring of the category of sheaves of modules on a topological space. See [1].

The Möbius algebra for a lower semilattice is just the semigroup algebra with respect to infimum (i.e. $x \cdot y = x \wedge y$) so that the Möbius algebra generalizes the semigroup algebra via Möbius inversion to arbitrary lower finite ordered sets. As such the Möbius algebra is an intersection algebra much like the Chow ring. Now it is known by the work of Grothendieck [12] and Baum-Fulton-MacPherson [5], that the Riemann-Roch theorem can be recast in a very general context as asserting that there is a natural transformation between the Chow ring and the Grothendieck ring. It was to find a purely combinatorial version of this result that motivated our work.

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2. Ordered Sets and Diagrams

We use the same notation for ordered sets and diagrams on ordered sets as in [4]. If P is an ordered set and $x \in P$, then V_x denotes $\{y \in P \mid y \geqslant x\}$. We define J_x dually. These are special cases of ascending and descending subsets of P respectively. We write 2^P for the ordered set (under inclusion) of ascending subsets of P. See Birkhoff [6]. The descending subsets of P form the closed sets of a topology on P. In this topology $\overline{\{x\}} = J_x$.

The ascending subsets of P of the form V_x are called the *principal* ascending subsets. An order-preserving map $f: P \to Q$ is said to be (*upper*) *Galois* if for every $y \in Q$, the subset $f^{-1}(V_y)$ of P is principal. Following Quillen [13], we abbreviate $f^{-1}(V_y)$ to simply f/y and call it the *fiber of f over y*. The ordered set obtained by reversing the order relation of P is called the *order dual* of P and is denoted P^*.

The relationship between Galois maps and Galois connections is given by Everett's theorem [9] which may be stated as follows:

PROPOSITION 1 (Everett). *Let $f: P \to Q$ be an order-preserving map of ordered sets.*

(1) *If for every $y \in Q$, the fiber f/y is either principal or empty, then there is a descending subset $J \subseteq Q$ such that the restriction $f: P \to J$ is Galois and conversely. Moreover J is unique if it exists.*

(2) *f is Galois if and only if there is an order-preserving map $g: Q \to P$ such that the pair (f, g) is a Galois connection between P^* and Q. Moreover g is unique if it exists.*

A *diagram on P with values in a category $\mathscr{C}$* is a commutative diagram whose underlying pattern is P. The class of all diagrams on P with values in $\mathscr{C}$ forms a category denoted $\mathscr{C}^P$. The value $D(x)$ of a diagram at a point $x \in P$ is called its *stalk* at x, also denoted D_x. We generally assume $\mathscr{C}$ is the category of modules over a ring R. The *support* $\mid D \mid$ of a diagram D is the subset of P on which D has nonzero stalks. The descending subset $J(D)$ generated by the support of D is called the *closed support* of D, $J(D) = \{x \in P \mid$ for some $y \geqslant x$, $D_y \neq 0\}$. A diagram D is said to be *finite* if $J(D)$ is finite and if all its stalks are finite modules over R.

An important diagram on an ordered set P is the *constant diagram* $\tilde{M}$, all of whose stalks are the module M over R and all of whose structure morphisms are the identity. There are two ways we may restrict diagrams. For a subset $S \subseteq P$, $D \mid S$ is the diagram on S given by the restriction to S. If S is *convex* (i.e., if for any $x, y \in S$, $\{z \mid x \leqslant z \leqslant y\} \subseteq S$), then we may define a diagram $D[S]$ whose stalks outside S have been set to zero but which coincides with D on S. For example, $\tilde{M}[\{x\}]$ is a diagram having a single nonzero stalk at x; we will abbreviate this to $M[x]$.

For any order-preserving map $f: P \to Q$, there is a functor $f^*: \mathscr{C}^Q \to \mathscr{C}^P$ given by $f^*E(x) = E(f(x))$, i.e. thinking of E as a functor $E: Q \to \mathscr{C}$, f^*E is $E \circ f$. We call f^* the *pullback*. The pullback has a right adjoint called the *pushout* $f_*: \mathscr{C}^P \to \mathscr{C}^Q$. To give an explicit description of f_*D we must extend the concept of the value of D to subsets of P. For $S \subseteq P$, the *value* $D(S)$ of D is the submodule of $\prod_{x \in S} D(x)$ consisting of *compatible* S-tuples, i.e. $(a_x \mid x \in S)$ is in $D(S)$ if the structure morphisms of D carry $a_x \mapsto a_y$ whenever $x \leqslant y$ in S. Then f_*D is given by $(f_*D)(y) = D(f/y)$.

The functor $f^*: \mathscr{C}^Q \to \mathscr{C}^P$ is exact, but $f_*: \mathscr{C}^P \to \mathscr{C}^Q$ is not, being only left exact. As a result, f_* gives rise to *higher direct images*, denoted $R^n f_*D$ (see Artin [2]). For the special case of Q being a single point, $R^n f_*D$ is the *cohomology* of D, denoted $H^n(P, D)$. In general, $(R^n f_*D)(y) = H^n(f/y, D)$. See [4] for explicit complexes to compute the cohomology. From these complexes it is easy to see that for D finite $R^n f_*D$ is also finite and furthermore that $R^n f_*D = 0$ for n sufficiently large. The connection of the higher direct images with Galois maps is provided by.

THEOREM 2. *If $f: P \to Q$ is an order-preserving map such that $f: P \to J$ is Galois for some descending subset $J \subseteq Q$, then $R^n f_*D = 0$ for all diagrams D on P, and for all n, $H^n(P, D) \cong H^n(Q, f_*D)$.*

For a proof see [4, Corollary 4.7].

The last result we need is the connection between the cohomology of diagrams and the Möbius functions of ordered sets. For this we use the following abbreviation. If P is an ordered set let $P_{\hat{0}}$ denote the ordered set obtained by adjoining a minimum element $\hat{0}$ to P, whether or not P has a minimum already. We then write $\mu_P(x) = \mu(\hat{0}, x)$, where $\mu(\hat{0}, x)$ is computed in $P_{\hat{0}}$.

PROPOSITION 3. *Let M be a finite module over a P.I.D. R, and let P be a lower finite ordered set. Then for any $x \in P$, we have*

$$\chi(M[x]) = -\mathrm{rank}_R(M) \cdot \mu_P(x),$$

where $\chi(M(x)) = \sum_{n=0}^{\infty} (-1)^n \mathrm{rank}_R H^n(P, M[x])$.

For a proof see [4, Lemma 5.1].

3. THE MÖBIUS ALGEBRA

Let P be a lower finite ordered set. Let $A_\wedge(P)$ be the free abelian group on P as a basis with product defined on basis elements by

$$x \cdot y = \sum_s \left(\sum_{\substack{s \leqslant t \leqslant x \\ s \leqslant t \leqslant y}} \mu(s, t) \right) s,$$

where μ is the Möbius function of P. We call $A_\wedge(P)$ the *Möbius algebra* of P (with coefficients in $\mathbb{Z}$). The Möbius algebra with coefficients in a ring R is $A_\wedge(P, R) = A_\wedge(P) \otimes_{\mathbb{Z}} R$. This is the original definition of Solomon [16].

Following Greene [11] we define the elements

$$\delta_x = \sum_{y \leqslant x} \mu(y, x) y, \qquad \text{for} \quad x \in P.$$

The δ_x are a complete set of orthogonal idempotents for $A_\wedge(P)$. Using these idempotents one can show Davis' theorem [8] that the Möbius algebra is isomorphic to a valuation ring.

More precisely, for a finite distributive lattice D let $V(D, R)$ be the valuation ring (Rota [15]) of D with coefficients in R. Then for a finite ordered set P

$$A_\wedge(P, R) \cong V(2^{P^*}, R)/(z),$$

where $z \in V(2^{P^*}, R)$ corresponds to the empty descending subset of P.

Now $A_\wedge(\cdot, R)$ is a covariant functor if for an order-preserving map $f: P \to Q$ we define $f_*: A_\wedge(P, R) \to A_\wedge(Q, R)$ on generators by $f_*(x) = f(x)$. However, as was noted by Greene [11, Theorem 2], f_* need not be a ring homomorphism. Indeed, he showed that f_* is a ring homomorphism essentially only when f is upper Galois. Similarly, as Geissinger [10] has noted, $A_\wedge(\cdot, R)$ is a contravariant functor if we define $f^*: A_\wedge(Q, R) \to A_\wedge(P, R)$ by

$$f^*(\delta_y) = \begin{cases} \sum\limits_{x \in f^{-1}(y)} \delta_x & \text{if} \quad f^{-1}(y) \neq \varnothing \\ 0 & \text{if} \quad f^{-1}(y) = \varnothing. \end{cases}$$

Of course, for this to be well-defined we must have that $f^{-1}(y)$ is finite for all $y \in Q$. We will say f is *finite* when this is the case.

4. The Grothendieck Ring

Let P be an ordered set and R a P.I.D. We write $\mathscr{FD}(P, R)$ for the category of finite diagrams on P. The isomorphism classes of $\mathscr{FD}(P, R)$ form a semiring $FD(P, R)$ under the operations of direct sum and tensor product (over R). We write $[D]$ for the isomorphism class of a finite diagram D.

Let $SFD(P, R)$ be the free abelian group generated by the elements of $FD(P, R)$. The semiring structure on $FD(P, R)$ extends to a ring structure on $SFD(P, R)$ which is commutative but does not have an identity element when P is infinite. Let $EFD(P, R)$ be the subgroup of $SFD(P, R)$ generated by elements

of the form $[D] - [E] + [F]$ where D, E, F are diagrams for which there is a short exact sequence

$$0 \to D \to E \to F \to 0.$$

We write $K(P, R)$ for the quotient $SFD(P, R)/EFD(P, R)$ and call it the *Grothendieck group* of $\mathscr{F}\mathscr{D}(P, R)$. There is a natural group homomorphism

$$FD(P, R) \to SFD(P, R) \to SFD(P, R)/EFD(P, R) = K(P, R)$$

which is denoted $\iota: FD(P, R) \to K(P, R)$. The image of a class $[D]$ in $FD(P, R)$ under ι will also be denoted $[D]$. The context should make it clear which use of "$[D]$" is intended.

The functorial properties of K are derived from the pullback and pushout functors of section 2. Let $f: P \to Q$ be order-preserving. Although f defines functors $f^*: \mathscr{C}^Q \to \mathscr{C}^P$ and $f_*: \mathscr{C}^P \to \mathscr{C}^Q$, only f_* restricts to a functor $f_*: \mathscr{F}\mathscr{D}(P, R) \to \mathscr{F}\mathscr{D}(Q, R)$. The functor f^* restricts to a functor $f^*: \mathscr{F}\mathscr{D}(Q, R) \to \mathscr{F}\mathscr{D}(P, R)$ if f is finite.

When f is finite $f^*: FD(Q, R) \to FD(P, R)$ is a semiring homomorphism. Since f^* obviously preserves exact sequences of diagrams, f^* induces a homomorphism $f^!: K(Q, R) \to K(P, R)$. In this way $K(\cdot, R)$ defines a contravariant functor from the category of lower finite ordered sets and finite maps to abelian groups.

Now for arbitrary $f: P \to Q$, f_* defines a semiring homomorphism $f_*: FD(P, R) \to FD(Q, R)$. However f_* need not preserve exact sequences so there is no reason to expect f_* to induce a homomorphism on the Grothendieck rings. The Leray spectral sequence however suggests an alternative approach. Namely define

$$f^![D] = \sum_{p=0}^{\infty} (-1)^p [R^p f_* D].$$

Note that if D is a finite diagram, then this is a finite sum. It is an immediate consequence of the long exact sequence for diagram cohomology that $f_!: K(P, R) \to K(Q, R)$ is a well-defined group homomorphism. The Leray spectral sequence then implies that $f_!$ endows $K(\cdot, R)$ with the structure of a covariant functor from the category of lower finite ordered sets to abelian groups.

5. THE ISOMORPHISM THEOREM

Throughout this section all ordered sets are lower finite and R is a P.I.D. We study the relationship between the functors $P \mapsto A_\Lambda(P)$ and $P \mapsto K(P, R)$.

We regard each as a functor from lower finite ordered sets to groups, having a covariant structure as well as the structure of a contravariant functor on finite maps.

LEMMA 5. *Let P be a finite ordered set. Define the diagram $\tilde{R}[V_x]$ for $x \in P$ as in section 2. Then*

(1) *the elements $[\tilde{R}[V_x]]$ generate $K(P, R)$,*

(2) *the elements $[R[x]]$ generate $K(P, R)$.*

Proof. Let D be a finite diagram on P. Let Q be the ascending subset of P generated by the support of D. Let x be a minimal element of Q. The stalk D_x is a finite R-module, so there is a surjective R-homomorphism $R^n \to D_x$. Since R is a P.I.D., the kernel of this homomorphism is isomorphic to a free R-module R^m. By the choice of x, this homomorphism extends uniquely to a morphism of diagrams $\alpha: (\tilde{R}[V_x])^{\oplus n} \to D$. Let $E = \mathrm{Ker}\,\alpha$ and $F = \mathrm{Coker}\,\alpha$.

The stalk of E at x is R^m so as above there is a unique morphism of diagrams $\beta: (\tilde{R}[V_x])^{\oplus m} \to E$. Set $E' = \mathrm{Ker}\,\beta$ and $F' = \mathrm{Coker}\,\beta$. By the definition of E and F, $[D] = n[\tilde{R}[V_x]] + [F] - [E]$. By the definition of E' and F', $[E] = m[\tilde{R}[V_x]] + [F'] - [E']$ so $[D] = (n - m)[\tilde{R}[V_x]] + [F] - [F'] + [E']$. Finally, the stalks of F, E' and F' at x are all zero. Because the elements $[D] \in K(P, R)$ for $D \in \mathscr{FD}(P, R)$ generate $K(P, R)$, we get (1) by the obvious induction.

The second statement is immediate from the first and the exact sequence $0 \to \tilde{R}[V_x - \{x\}] \to \tilde{R}[V_x] \to R[x] \to 0$, for $x \in P$, by the same induction.

Q.E.D.

In the special case $P = \{\hat{0}\}$, both $A_\Lambda(\{\hat{0}\})$ and $K(\{\hat{0}\}, R)$ are isomorphic to $\mathbb{Z}$.

PROPOSITION 6. *Let $\eta: K(\{\hat{0}\}, R) \to A_\Lambda(\{\hat{0}\})$ be a homomorphism of groups. Then there is at most one natural transformation $\alpha: K(\cdot, R) \to A_\Lambda$, where both are regarded as covariant functors, such that $\alpha(\{\hat{0}\}) = \eta$.*

Proof. The proof proceeds in steps. We first consider finite chains. Let C_n be the chain $\{0, 1, ..., n\}$ of length n. For $m \in C_n$, $m' \in C_{n-1}$ define order-preserving maps $g_m : C_{n-1} \to C_n$ and $h_{m'} : C_n \to C_{n-1}$ by:

$$g_m(l) = \begin{cases} l & \text{if } l < m \\ l + 1 & \text{if } l \geqslant m \end{cases} \quad \text{and} \quad h_{m'}(l) = \begin{cases} l & \text{if } l \leqslant m' \\ l - 1 & \text{if } l > m' \end{cases}$$

We may assume that $n \geqslant 1$. Then for $m \in C_n$,

$$\begin{array}{ccc} K(C_{n-1}, R) & \xrightarrow{(g_m)_!} & K(C_n, R) \\ \downarrow{\scriptstyle\alpha(C_{n-1})} & & \downarrow{\scriptstyle\alpha(C_n)} \\ A_\Lambda(C_{n-1}) & \xrightarrow{(g_m)_*} & A_\Lambda(C_n) \end{array}$$

commutes. Now g_m is a Galois map. So the higher direct images $R^p g_m$ all vanish and $(g_m)_![D] = [(g_m)_* D]$ for any diagram D on P. So for $0 \leqslant l \leqslant n - 1$,

$$(g_m)_![\tilde{R}[V_l]] = [(g_m)_* \tilde{R}[V_l]] = \begin{cases} [\tilde{R}[V_l]] & \text{if } l \leqslant m \\ [\tilde{R}[V_{l+1}]] & \text{if } l > m. \end{cases}$$

Accordingly if $1 \leqslant l \leqslant n - 1$, then

$$\begin{aligned} \alpha(C_n)[\tilde{R}[V_l]] &= \alpha(C_n) \circ (g_n)_![\tilde{R}[V_l]] \\ &= (g_n)_* \circ \alpha(C_{n-1})[\tilde{R}[V_l]], \end{aligned}$$

and if $l = n$, then

$$\begin{aligned} \alpha(C_n)[\tilde{R}[V_l]] &= \alpha(C_n) \circ (g_1)_![\tilde{R}[V_{l-1}]] \\ &= (g_1)_* \circ \alpha(C_{n-1})[\tilde{R}[V_{l-1}]]. \end{aligned}$$

Therefore $\alpha(C_n)$ is determined by $\alpha(C_{n-1})$, by Lemma 5. By induction the value of α on chains is determined by $\alpha(\{\hat{0}\})$.

Now let P be an arbitrary finite ordered set. By Szipilrajn's Theorem [17], there is a bijective order-preserving map $f: P \to C_n$, where $n + 1$ is the cardinality of P. By Greene [11, Theorem 1], $f_*: A_\Lambda(P) \to A_\Lambda(C_n)$ is an isomorphism of groups. Therefore

$$\alpha(P) = (f_*)^{-1} \circ \alpha(C_n) \circ f_! : K(P, R) \to A_\Lambda(P).$$

So $\alpha(P)$ is determined on arbitrary finite ordered sets by $\alpha(\{\hat{0}\})$.

Finally let P be an arbitrary lower finite ordered set. Let D be a finite diagram on P. Let Q be the closed support of D. Then Q is a finite descending subset of P. Let $i: Q \to P$ be the inclusion. Then i is Galois. So writing $E = i^*D$ and noticing that $i_* i^* D = D$ in this case, we have

$$\alpha(P)[D] = \alpha(P)[i_* E] = \alpha(P) \circ i_![E] = i_* \circ \alpha(Q)[E].$$

Since Q is finite, we conclude that α is determined in general by $\alpha(\{\hat{0}\})$. Q.E.D.

THEOREM 7. *There is a unique natural isomorphism* $\alpha: K(\cdot, R) \to A_\Lambda$, *which is natural with respect to both the covariant and contravariant functors, such that* $\alpha(\{\hat{0}\}) = \mathrm{id}_{\mathbb{Z}} : K(\{\hat{0}\}, R) \to A_\Lambda(\{\hat{0}\})$.

Proof. Let P be a lower finite ordered set. Define a group homomorphism $\gamma(P): A_\Lambda(P) \to K(P, R)$ by $\gamma(P)(\delta_x) = [R[x]]$ where δ_x was defined in section 3 and where $x \in P$. We show naturality and bijectivity in steps.

We first show γ is natural with respect to the contravariant structure on the subcategory of lower finite ordered sets and injective maps. Let $f: P \to Q$ be in this subcategory. For $y \in Q$, $f^{-1}(y)$ is either empty or a singleton. So $f^*R[y]$ is either 0 or $R[f^{-1}(y)]$. Hence for $y \in Q$

$$\gamma(P) \circ f^*(\delta_y) = \begin{cases} \gamma(P)(\delta_{f^{-1}(y)}) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing \end{cases}$$

$$= \begin{cases} [R[f^{-1}(y)]] & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing \end{cases}$$

$$= [f^*R[y]]$$

$$= f^![R[y]]$$

$$= f^! \circ \gamma(Q)(\delta_y).$$

Therefore γ is natural on this subcategory.

We show $\gamma(P)$ is injective. Let $\sum \eta_x \delta_x$ be an element of $A_\wedge(P)$ so that all but finitely many $\eta_x = 0$. Suppose $\gamma(P)(\sum \eta_x \delta_x) = \sum \eta_x[R[x]]$ vanishes in $K(P, R)$. Let $x \in P$ and let $h: \{x\} \to P$ be the inclusion. Then

$$0 = h^! \circ \gamma(P)\left(\sum \eta_x \delta_x\right)$$

$$= \gamma(\{y\}) \circ h^*\left(\sum \eta_x \delta_x\right)$$

$$= \gamma(\{y\})(\eta_y \delta_y)$$

$$= \eta_y[R[y]].$$

Now $K(\{y\}, R) \cong \mathbb{Z}$ and $[R[y]]$ is the identity element. Therefore $\eta_y = 0$. It follows that $\sum \eta_x \delta_x = 0$ and that $\gamma(P)$ is injective.

Now we check surjectivity. First suppose $i: Q \to P$ is the inclusion of a finite descending subset of P and that $y \in Q$. Then

$$i_*(\delta_y) = i_*\left(\sum_{x \leqslant y} \mu_Q(x, y)x\right)$$

$$= \sum_{x \leqslant y} \mu_Q(x, y)x$$

$$= \sum_{x \leqslant y} \mu_P(x, y)x \qquad \text{since } Q \text{ is descending}$$

$$= \delta_{i(y)}.$$

Now if $\sum \eta_y \delta_y$ is in $A_\wedge(Q)$,

$$i_! \circ \gamma(Q) \left(\sum \eta_y \delta_y \right) = i_! \left(\sum \eta_y [R[y]] \right)$$

$$= \sum \eta_y [i_* R[y]] \qquad \text{by Theorem 2}$$

$$= \sum \eta_y [R[i(y)]] \qquad \text{since } Q \text{ is descending}$$

$$= \sum \eta_y \gamma(P)(\delta_{i(y)})$$

$$= \gamma(P) \circ i_* \left(\sum \eta_y \delta_y \right) \qquad \text{by the above computation.}$$

Therefore $i_! \circ \gamma(Q) = \gamma(P) \circ i_*$. Let D be a finite diagram on P. Write Q for $J(D)$ and $i: Q \to P$ for the inclusion. Set $E = D \mid Q = i^*D$, so that $i_![E] = [D]$. We already know that $\gamma(Q)$ is bijective by Lemma 5, because Q is finite. Let $a \in A_\wedge(Q)$ be such that $\gamma(Q)(a) = [E]$, then $[D] = i_! \circ \gamma(Q)(a) = \gamma(P)(i_*(a))$. It follows that $\gamma(P)$ is surjective.

We may therefore define $\alpha(P) = \gamma(P)^{-1} \colon K(P, R) \to A_\wedge(P)$ for any lower finite ordered set P. Then α defines a natural isomorphism $\alpha \colon K \to A_\wedge$ on the subcategory of injective maps with respect to the contravariant structures. We compute $\alpha(P)$ explicitly.

For a finite diagram D, suppose that $\alpha(P)[D]$ is $\sum \eta_x \delta_x$. Let y be in P, and let $h \colon \{y\} \to P$ be the inclusion. Then

$$\eta_y \delta_y = \sum \eta_x h^*(\delta_x)$$

$$= h^* \left(\sum \eta_x \delta_x \right)$$

$$= h^* \circ \alpha(P)[D]$$

$$= \alpha(\{y\}) \circ h^![D]$$

$$= \alpha(\{y\})[D_y[y]]$$

$$= (\text{rank}_R D_y) \, \delta_y .$$

Therefore, $\eta_y = \text{rank}_R D_y$ and

$$\alpha(P)[D] = \sum_{x \in P} (\text{rank}_R D_x) \delta_x .$$

We have already shown that α is unique in Proposition 6, so it remains to show that α is natural for all maps and both functorial structures. Let $f \colon P \to Q$ be order-preserving. First the contravariant case. For $y \in Q$,

$$\alpha(P) \circ f^!\,[R[y]] = \alpha(P)[f*R[y]]$$

$$= \sum_{x \in P} \operatorname{rank}_R((f*R[y])_x)\delta_x$$

$$= \sum_{x \in P} \operatorname{rank}_R(R[y]_{f(x)})\delta_x$$

$$= \sum_{x \in f^{-1}(y)} \delta_x$$

$$= f*(\delta_y)$$

$$= f* \circ \alpha(Q)[R[y]].$$

Now the covariant case. For $x \in P$,

$$\alpha(Q) \circ f_!\,[R[x]] = \alpha(Q) \sum_{n=0}^{\infty} (-1)^n [R^n f_* R[x]]$$

$$= \sum_{n=0}^{\infty} (-1)^n \sum_{y \in Q} \operatorname{rank}_R((R^n f_* R[x])_y)\delta_y$$

$$= \sum_{n=0}^{\infty} (-1)^n \sum_{y \in Q} \operatorname{rank}_R H^n(f/y, R[x])\delta_y$$

$$= \sum_{y \in Q} \left(\sum_{n=0}^{\infty} \operatorname{rank}_R H^n(f/y, R[x]) \right) \delta_y$$

$$= \sum_{\substack{y \in Q \\ y \leqslant f(x)}} \chi(f/y, R[x])\delta_y$$

$$= \sum_{\substack{y \in Q \\ y \leqslant f(x)}} -\mu_{(f/y)}(x)\delta_y \qquad\qquad\qquad \text{by Proposition 3}$$

$$= \sum_{\substack{y \in Q \\ y \leqslant f(x)}} -\mu_{(f/y)}(x) \left(\sum_{z \leqslant y} \mu_Q(z, y)z \right)$$

$$= \sum_{\substack{y \in Q \\ y \leqslant f(x)}} \left(\sum_{\substack{w \in f/y \\ w \leqslant x}} \mu_P(w, x) \right) \left(\sum_{z \leqslant y} \mu_Q(z, y)z \right)$$

by the definition of μ on P and the fact that $f^{-1}(V_y)$ is an ascending subset.

$$= \sum_{\substack{y \in Q \\ y \leqslant f(x)}} \sum_{z \leqslant y} \left(\sum_{\substack{f(w) \geqslant y \\ w \leqslant x}} \mu_P(w, x)\, \mu_Q(z, y) \right) z$$

$$= \sum_{z \in Q} \left(\sum_{w \leqslant x} \left(\sum_{z \leqslant y \leqslant f(w)} \mu_Q(z, y)\, \mu_P(w, x) \right) z \right)$$

$$= \sum_{z \in Q} \sum_{\substack{w \leqslant x \\ z = f(w)}} \mu_P(w, x) z \qquad\qquad \text{by the definition of } \mu \text{ on } Q$$

$$= \sum_{w \leqslant x} \mu_P(w, x) f_*(w)$$

$$= f_* \left(\sum_{w \leqslant x} \mu_P(w, x)_w \right)$$

$$= f_*(\delta_x)$$

$$= f_* \circ \alpha(P)[R[x]]. \qquad\qquad\qquad\qquad\qquad \text{Q.E.D.}$$

6. Consequences

Several results follow immediately from Theorem 7.

COROLLARY 8.   *The subgroup $EFD(P, R)$ is an ideal of $SFD(P, R)$ and $K(P, R)$ is a commutative ring (not necessarily with 1). Moreover α is a natural isomorphism of rings.*

Proof.   That $K(P, R)$ is a ring can be shown directly. The real point here is to see that α endows $K(P, R)$ (from the ring structure on $A_\wedge(P)$) with the ring structure we expect from the tensor product. That is, we wish to show that $[D \otimes_R E] = [D][E]$, where the product on the right is the one defined by α. This result follows immediately from the explicit computations of the last section and from the fact that $\mathrm{rank}_R(M \otimes_R N) = (\mathrm{rank}_R M)(\mathrm{rank}_R N)$ for any two finite R-modules M and N. $\qquad\qquad\qquad$ Q.E.D.

COROLLARY 9 (Greene).   *For an order-preserving map $f: P \to Q$, f_* defines a ring homomorphism $f_* : A_\wedge(P) \to A_\wedge(Q)$ if and only if $f: P \to J$ is Galois for some descending subset $J \subseteq Q$. When P and Q are finite, f preserves the identity elements if and only if f is upper Galois.*

Proof.   We prove the result by proving the corresponding result for $f_! :$ $K(P, R) \to K(Q, R)$. Let f satisfy the condition of the first statement. By Theorem 2, the $R^p f_* D$ vanish for $p > 0$. So we need only show that $[f_*(D \otimes_R E)] = [f_*D][f_*E]$, for diagrams D, E on P. Let y be in Q. Then either f/y is empty or $f/y = V_x$ for some $x \in P$. When $f/y = V_x$, $(f_*(D \otimes_R E))_y$ $= (D \otimes_R E)_x = D_x \otimes_R E_x = (f_*D)_y \otimes_R (f_*E)_y$. When $f/y = \varnothing$, $(f_*(D \otimes_R E))_y$ $= 0 = (f_*D)_y \otimes_R (f_*E)_y$. Therefore $f_*(D \otimes_R E) = (f_*D) \otimes_R (f_*E)$.

Conversely suppose $f_!$ is a ring homomorphism. By Theorem 4, the semiring homomorphism $\pi: FD(P, R) \to \mathbb{Z}$ given by $\pi[D] = \dim_R D_y$ induces a surjective ring homomorphism $\pi: K(P, R) \to \mathbb{Z}$.

Suppose f does not satisfy the condition. Then for some $y \in Q$, f/y has at least two minimal elements, say x and x'. We then compute

$$(R^p f_* R[x])_y = H^p(f/y, R[x]) = \begin{cases} R & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Similarly for $(R^p f_* R[x'])_y$. Therefore $\pi(f_!{[R[x]]}) = 1$ and $\pi(f_!{[R[x']]}) = 1$. But $R[x] \otimes_R R[x'] = 0$. So we have a contradiction. The first statement then follows.

If P and Q are finite, we see that $f_!$ need not preserve the identity element even though both $K(P, R)$ and $K(Q, R)$ have one. The identity element is $[\tilde{R}]$ in each case, and $f_* \tilde{R} = \tilde{R}$ precisely when f is a Galois map by an obvious computation. It is easy to see that $f_![\tilde{R}] = [\tilde{R}]$ in just this case also. This gives the second statement and the Corollary.   Q.E.D.

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