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Whitney Numbers of Geometric Lattices

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INTRODUCTION

It has been a long-standing conjecture of Rota [14] that there is a homology theory on the category of ordered sets such that the Betti numbers of a geometric lattice are the Whitney numbers of the first kind. The purpose of this paper is to describe such a theory. We will also show that our theory and the usual simplicial theory are related by a spectral sequence.

The theory we develop is a sheaf cohomology on a topological space associated to the ordered set. It is however, also possible to develop the theory in terms of specific simplicial chain complexes. For those who find sheaves unpalatable, we describe these chain complexes in Section 5.

In Section 1, we develop some sheaf theory for the special case in which we will be dealing. In the next section, we examine the cohomology theories of two sheaves: the sheaf of locally constant integer-valued functions and the sheaf $\mathscr{M}$, based on the "valuation ring" of Rota [11]. The former sheaf, of course, gives the ordinary simplicial cohomology theory, while $\mathscr{M}$ gives the same theory in dimensions greater than zero when the ordered set is finite. In Section 3, we define a sheaf $\mathscr{W}$ whose cohomology groups are groups whose ranks are the Whitney numbers when the ordered set (after adjoining a zero) is a geometric lattice. In the next section, we describe a spectral sequence that relates the cohomology theory of $\mathscr{W}$ to the ordinary simplicial cohomology theory. In the last section, we describe a way to relate "transitive" elements of the incidence algebra of the ordered set to certain sheaves on the ordered set. Simplicial chain complexes that give the cohomology of these sheaves are then

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described, giving an alternative approach to the theories developed in the earlier sections.

The idea of using sheaf theory for studying ordered sets is not new; see, for example, Graves and Molnar [9]. It was their work that inspired this paper. The author wishes to thank Prof. Rota and Prof. Graves for many discussions on this subject.

1. PRELIMINARIES

Let P be an ordered set. An *increasing subset* (or *order-filter*) of P is a subset $U \subseteq P$ such that $x \in U$ and $y \geqslant x$ imply that $y \in U$. One similarly defines *decreasing subset* (or *order-ideal*). The increasing subsets of P are easily seen to be the open sets for a topology on P. In the sequel, we always assume that ordered sets are endowed with this topology. The order-preserving maps from P to another ordered set, Q, are precisely the continuous functions from P to Q. Every point $x \in P$ is contained in a unique smallest open set:

$$V_x = \{y \in P \mid y \geqslant x\}$$

called the *principal filter* of x. See Graves and Molnar [9].

We may also regard P as a category. The objects of this category are the elements of P, and the morphisms are relations of the form $x \leqslant y$, the source of this morphism being x and the target being y. Let $\mathscr{C}$ be the category of abelian groups and homomorphisms. A *sheaf* on P is a covariant functor $\mathscr{F}: P \to \mathscr{C}$. The value of $\mathscr{F}$ at $x \in P$ is a group called the *stalk* of $\mathscr{F}$ at x and will be written $\mathscr{F}(V_x)$. For $x \leqslant y$ in P, the corresponding homomorphism $\mathscr{F}(V_x) \to \mathscr{F}(V_y)$ is called the *restriction* from V_x to V_y. Let $U \subseteq P$ be an open subset, and let U^* be the ordered set obtained by reversing the directions of the inequalities in U (i.e., $x \leqslant y$ in U^* if and only if $x \geqslant y$ in U). Then the restriction of $\mathscr{F}$ to U is an *inverse system* (in the general sense) on U^*. We write $\mathscr{F}(U)$ for the inverse limit $\varprojlim_{x \in U^*} \mathscr{F}(V_x)$. This notation is easily seen to be consistant with the notation $\mathscr{F}(V_x)$ used for the stalks. It is easy to see that $\mathscr{F}$, as a functor on the category of open subsets and inclusions of P, is a sheaf on P in the usual sense of the term.

A sheaf $\mathscr{G}$ on P is said to be *flasque* if, for any open set $U \subseteq P$, the restriction $\mathscr{G}(P) \to \mathscr{G}(U)$ is surjective. Let $\mathscr{F}$ be a sheaf on P. A *flasque resolution* of $\mathscr{F}$ is an exact sequence:

$$0 \to \mathscr{F} \to \mathbf{C}^0 \to \mathbf{C}^1 \to \mathbf{C}^2 \to \cdots$$

of sheaves on P such that $\mathbf{C}^i$ is flasque for all $i \geqslant 0$. Replacing each of these sheaves by the group $\mathbf{C}^i(P)$, defines a complex

$$0 \to \mathbf{C}^0(P) \to \mathbf{C}^1(P) \to \mathbf{C}^2(P) \to \cdots$$

of groups, whose cohomology is called the *cohomology of P with coefficients in the sheaf $\mathscr{F}$* and is denoted $H^i(P, \mathscr{F})$. We write $H^*(P, \mathscr{F})$ for the (graded) direct sum of the $H^i(P, \mathscr{F})$. One can show that the cohomology is independent, up to a natural isomorphism, of the flasque resolution used. See Godement [8, II.4.7.1].

Let $\mathscr{F}$ be a sheaf on P. The *sheaf of discontinuous sections* of $\mathscr{F}$ is the sheaf $[\mathscr{F}]$ defined on each open $U \subseteq P$ by

$$[\mathscr{F}](U) = \prod_{x \in P} \mathscr{F}(V_x),$$

with restrictions given by the obvious projection homomorphisms. Clearly, $[\mathscr{F}]$ is a flasque sheaf, and the canonical morphism $\epsilon\colon \mathscr{F} \to [\mathscr{F}]$ induced by the restrictions $\mathscr{F}(U) \to \mathscr{F}(V_x)$ is an injective morphism of sheaves. The inductively defined sequence of sheaves:

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} [\mathscr{F}] \xrightarrow{d_0} [\mathrm{Coker}(\epsilon)] \xrightarrow{d_1} [\mathrm{Coker}(d_0)] \longrightarrow \cdots$$

gives a flasque resolution of $\mathscr{F}$ called the *canonical resolution* of $\mathscr{F}$. Thus every sheaf on P has a flasque resolution, and sheaf cohomology is defined for every sheaf on P.

For the basic properties of sheaves and sheaf cohomology, see Godement [8].

Let P and Q be ordered sets and $f\colon P \to Q$ an order-preserving map. Let $\mathscr{F}$ be a sheaf on P. The *direct image sheaf* of $\mathscr{F}$ on Q is the sheaf defined by: $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$ for open subsets U of P.

LEMMA 1.1. *Let P be a decreasing subset of Q and $i\colon P \to Q$ the inclusion. Then, for any sheaf $\mathscr{F}$ on P, there is a natural isomorphism:*

$$H^*(Q, i_*\mathscr{F}) \cong H^*(P, \mathscr{F}).$$

Proof. Since P is a descending subset of Q, if $x \in Q$ and $V_x \cap P \neq \varnothing$, then $x \in P$. Now $i_*[\mathscr{F}](V_x) = [\mathscr{F}](V_x \cap P) = \prod_{y \in V_x \cap P} \mathscr{F}(V_y \cap P)$, and $[i_*\mathscr{F}](V_x) = \prod_{y \in V_x} i_*\mathscr{F}(V_y) = \prod_{y \in V_x} \mathscr{F}(V_y \cap P)$. Since $V_y \cap P$ is the principal filter of y in P when $y \in P$ and since $V_y \cap P = \varnothing$ when $y \notin P$, these two products coincide. Therefore, $i_*[\mathscr{F}]$ and $[i_*\mathscr{F}]$ are naturally isomorphic. By an inductive procedure, we extend this isomorphism to an isomorphism of the canonical resolution of $i_*\mathscr{F}$ with the direct image of the canonical resolution of $\mathscr{F}$.

2. The Sheaves $\tilde{\mathbf{Z}}$ and $\mathscr{M}$

We define the *sheaf* $\tilde{\mathbf{Z}}$ *of locally constant integer-valued functions on* P by $\tilde{\mathbf{Z}}(V_x) = \mathbf{Z}$ for all $x \in P$, with the restrictions being the identity maps. More generally, for an arbitrary abelian group G, we define $\tilde{G}$ in a similar fashion.

We regard P as a simplicial complex in the usual fashion: the vertices are the elements of P, and the k-simplices are the totally ordered $k + 1$ element subsets of P. We denote the simplicial homology and cohomology of P with coefficients in the abelian group G by $H_*(P, G)$ and $H^*(P, G)$ respectively. As one would expect, the simplicial cohomology and the cohomology with coefficients in $\tilde{G}$ coincide.

THEOREM 2.1. *For any ordered set* P *and abelian group* G, *there is a natural isomorphism*:

$$H^*(P, \tilde{G}) \cong H^*(P, G).$$

Proof. A more general result is proved by Deheuvels [5, Section 11]. We will give a brief sketch of a proof.

Let $C_*(P, \mathbf{Z})$ be the simplicial chain complex of P. Let $C^*(P, G)$ be the dual of $C_*(P, \mathbf{Z})$, i.e., $\mathrm{Hom}(C_*(P, \mathbf{Z}), G)$. The cohomology of $C^*(P, G)$ is then the simplicial cohomology $H^*(P, G)$. Since $C^*(P, G)$ is functorial in P, we may define a complex of sheaves by $\mathbf{C}^*(P, G)(V_x) = C^*(V_x, G)$ for all $x \in P$. Moreover, there is a natural inclusion of sheaves $\tilde{G} \to \mathbf{C}^0(P, G)$ given by mapping $g \in G(V_x)$ to the element $\chi_g \in C^0(V_x, G)$ defined by $\chi_g(y) = g$ for all $y \in V_x$. Since the simplicial cohomology of a principal filter is trivial, the following is an exact sequence of sheaves on P:

$$0 \to \tilde{G} \to \mathbf{C}^0(P, G) \to \mathbf{C}^1(P, G) \to \cdots.$$

It is easily checked that the sheaves $\mathbf{C}^i(P, G)$ are all flasque. In fact, if we define a sheaf $\mathscr{G}_i$ by $\mathscr{G}_i(V_x) =$ the G-dual of the free abelian group generated by all i-simplices of P with minimal element x, then $\mathbf{C}^i(P, G) \cong [\mathscr{G}_i]$. Therefore, $\mathbf{C}^*(P, G)$ is a flasque resolution of $\tilde{G}$.

Since the group of global sections, $\mathbf{C}^i(P, G)(P)$, is just $C^i(P, G)$, for all i, the result follows.

In [11], Rota introduced an augmented ring $V(L)$ associated to every distributive lattice L, called the *valuation ring* of L. This ring is formed as follows. Let $F(L)$ be the free abelian group generated by L. We give this group a ring structure by defining the product of the basis elements $x, y \in L$ to be $x \wedge y$. Now let J be the ideal of $F(L)$ generated by elements of the form $x + y - x \wedge y - x \vee y$, for $x, y \in L$. The valuation ring of L is then the quotient ring $V(L) = F(L)/J$, adjoining an identity element if $F(L)/J$ does not already have one. We define a homomorphism of rings, $\epsilon: F(L) \to \mathbf{Z}$, by $\epsilon(x) = 1$ for all $x \in L$. Since $J \subseteq \operatorname{Ker}(\epsilon)$, ϵ induces a ring homomorphism $\epsilon: V(L) \to \mathbf{Z}$ which we call the *augmentation* of $V(L)$.

For an ordered set Q, let $L(Q)$ be the lattice of decreasing subsets of Q, and write $M(Q)$ for $V(L(Q))$. The functorial properties of L, V and M should be clear: V is covariant while both L and M are contravariant. Hence we may define a sheaf on P by $\mathscr{M}(V_x) = M(V_x)$, for all $x \in Q$, with the restriction $\mathscr{M}(V_x) \to \mathscr{M}(V_y)$, for $x \leqslant y$ in Q, being induced functorially by the inclusion $V_y \to V_x$. Note that, in general, $\mathscr{M}(Q) \neq M(Q)$. $\mathscr{M}$ is in fact a sheaf of augmented rings, but we do not use this.

An ordered set P is said to be *upper finite* [*lower finite*] if the principal filters V_x [principal ideals $J_x = \{y \in P \mid y \leqslant x\}$] are finite for all $x \in P$.

THEOREM 2.2. *If Q is an upper finite ordered set, then for $i > 0$:*

$$H^i(Q, \mathscr{M}) \cong H^i(Q, \mathbf{Z}).$$

Moreover, if Q is finite and connected, then $H^0(Q, \mathscr{M}) \cong M(Q)$.

Proof. Each stalk $\mathscr{M}(V_x)$ contains an element z_x which corresponds to the minimum element of the lattice $L(V_x)$. We map $\tilde{\mathbf{Z}}$ to $\mathscr{M}$ by sending $1 \in \tilde{\mathbf{Z}}(V_x) = \mathbf{Z}$ to $z_x \in \mathscr{M}(V_x)$ for all $x \in Q$. This map is easily seen to define an injective morphism of sheaves $\tilde{\imath}: \tilde{\mathbf{Z}} \to \mathscr{M}$. We may also map $\mathscr{M}$ to $\tilde{\mathbf{Z}}$ by mapping $\mathscr{M}(V_x) \to \tilde{\mathbf{Z}}(V_x) = \mathbf{Z}$ via the augmentation. This map defines a surjective morphism of sheaves, $\tilde{\epsilon}: \mathscr{M} \to \tilde{\mathbf{Z}}$. Clearly, the composition $\tilde{\epsilon} \circ \tilde{\imath}$ is the identity on $\tilde{\mathbf{Z}}$. Let $\bar{\mathscr{M}}$ be the kernel of $\tilde{\epsilon}$. Then $\bar{\mathscr{M}}$ coincides with the sheaf denoted M in Graves and Molnar [9].

We now prove that $\bar{\mathscr{M}} \cong [\tilde{\mathbf{Z}}]$, and hence that $\bar{\mathscr{M}}$ is flasque. It is a result of Davis [4] that $\bar{\mathscr{M}}(V_x) \cong M(V_x)/(z_x)$ is a free abelian group generated by a set of mutually orthogonal idempotents corresponding bijectively to the set V_x. Moreover, the restriction $\bar{\mathscr{M}}(V_x) \to \bar{\mathscr{M}}(V_y)$ for $x \leqslant y$ is given by the projection homomorphism of the first onto the second as a direct summand. Since for finite sets of abelian groups the direct sum and the direct product coincide, it follows that $\bar{\mathscr{M}} \simeq [\tilde{\mathbf{Z}}]$.

Now the morphism $\tilde{\imath}: \tilde{\mathbf{Z}} \to \mathscr{M}$ splits the short exact sequence of sheaves:

$$0 \longleftarrow \tilde{\mathbf{Z}} \overset{\tilde{\epsilon}}{\longleftarrow} \mathscr{M} \longleftarrow \bar{\mathscr{M}} \longleftarrow 0.$$

Since flasque sheaves are cohomologically trivial, the long exact sequence of sheaf cohomology gives the result, by Theorem 2.1.

We add that there are results corresponding to those above for cosheaves instead of sheaves. Such objects were studied extensively by Deheuvels [5]. We will just state the results briefly and without proof.

A *cosheaf* on an ordered set P is a contravairant functor $\mathscr{F}: P \to \mathscr{C}$. The value of $\mathscr{F}$ at $x \in P$ is called the *stalk* of $\mathscr{F}$ at x, and is denoted $\mathscr{F}(V_x)$. For $x \leqslant y$ in P, the homomorphism $\mathscr{F}(V_y) \to \mathscr{F}(V_x)$ is called the *extension* from V_y to V_x. For an open subset $U \subseteq P$, $\mathscr{F}(U)$ is the direct limit $\varinjlim_{x \in U^*} \mathscr{F}(V_x)$, and we call $\mathscr{F}(U)$ the *group of cosections* of $\mathscr{F}$ on U. A cosheaf $\mathscr{F}$ is said to be *flasque* if, for any open set $U \subseteq P$, the extension $\mathscr{F}(U) \to \mathscr{F}(P)$ is injective. A *flasque resolution* of the cosheaf $\mathscr{F}$ is an exact sequence:

$$\cdots \to \mathbf{C}^2 \to \mathbf{C}^1 \to \mathbf{C}^0 \to \mathscr{F} \to 0$$

of cosheaves on P such that $\mathbf{C}^i$ is flasque for all $i \geqslant 0$. As above, we may use flasque resolutions of cosheaves to define the *homology of P with coefficients in a cosheaf $\mathscr{F}$*, denoted $H_i(P, \mathscr{F})$.

We define the cosheaf $\tilde{\mathbf{Z}}'$ by $\tilde{\mathbf{Z}}'(V_x) = \mathbf{Z}$ for all $x \in P$, with the extensions all being the identity map. For an abelian group G, we define $\tilde{G}'$ similarly.

THEOREM 2.3. *For any ordered set P and abelian group G, there is a natural isomorphism*:

$$H_*(P, \tilde{G}') \simeq H_*(P, G).$$

For an ordered set Q, we define the *valuation cosheaf* $\mathscr{M}'$ by $\mathscr{M}'(V_x) = \mathrm{Hom}(M(V_x), \mathbf{Z})$, with the obvious extensions.

THEOREM 2.4. *If Q is any ordered set, then for $i > 0$*:

$$H_i(Q, \mathscr{M}') \simeq H_i(Q, \mathbf{Z}).$$

Moreover, if Q is connected, then $H^0(Q, \mathscr{M}') \simeq \mathrm{Hom}(M(Q), \mathbf{Z})$.

3. The Sheaf $\mathscr{W}$

Let P be a lower finite ordered set. We denote by $\hat{P}$ the ordered set obtained by adjoining a unique minimum element 0 to P (whether or not P already has a minimum element). Then $\hat{P}$ is locally finite so we may speak of the incidence algebra $I(\hat{P})$ of $\hat{P}$ (with coefficients in $\mathbf{Z}$) with zeta function ζ, Möbius function μ and identity element δ. See [6] or [12] for the definitions and terminology.

Let $x \in P$, and let G be an abelian group. We define the sheaf $G(x)$ on P by

$$G(x)(V_y) = \begin{cases} G & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

with the restrictions, of course, all being zero. We will be examining the cohomology of P with coefficients in the sheaf $G(x)$ in terms of the Möbius function on $\hat{P}$.

Lemma 3.1. *Let P be a lower finite ordered set, $x \in P$ and G an abelian group. Then*

$$H^0(P, G(x)) \cong \begin{cases} G & \textit{if } x \textit{ is a minimal element of } P, \\ 0 & \textit{if } x \textit{ is not minimal in } P, \end{cases}$$

and for $i > 0$

$$H^i(P, G(x)) \cong \tilde{H}^{i-1}((0, x), G).$$

Proof. Let η: $G(x) \to [G(x)]$ be the canonical inclusion of sheaves. We then get the short exact sequence of sheaves on P

$$0 \longrightarrow G(x) \xrightarrow{\eta} [G(x)] \longrightarrow \mathrm{Coker}(\eta) \longrightarrow 0.$$

Examining this sequence on stalks, one finds that $\mathrm{Coker}(\eta)$ is the sheaf of locally constant G-valued functions on the open interval $(0, x)$ in P. Hence, by Lemma 1.1 and Theorem 2.1, the cohomology of $\mathrm{Coker}(\eta)$ is just the simplicial cohomology of $(0, x)$ with coefficients in G.

Since $[G(x)]$ is flasque, the long exact sequence of the above short exact sequence of sheaves gives the result for $i > 1$. For $i = 0$ and 1, consider the first part of the long exact sequence:

$$0 \to H^0(P, G(x)) \to H^0(P, [G(x)]) \to H^0(P, \mathrm{Coker}(\eta)) \to H^1(P, G(x)) \to 0.$$

If x is minimal, then $(0, x) = \varnothing$; hence $H^0(P, \mathrm{Coker}(\eta)) = 0$. Thus $H^1(P, G(x)) = 0 = \tilde{H}^0((0, x), G)$, and $H^0(P, G(x)) \cong H^0(P, [G(x)]) \cong G$, in this case. If x is not minimal, then $H^0(P, \mathrm{Coker}(\eta))$ is isomorphic to a direct sum of copies of G, the number of copies being the number of connected components of $(0, x)$. Now the map $G \simeq H^0(P, [G(x)]) \to H^0(P, \mathrm{Coker}(\eta))$ is easily seen to be the inclusion of G as the diagonal of $H^0(P, \mathrm{Coker}(\eta))$. Hence the cokernel of this map is isomorphic to $\tilde{H}^0((0, x), G)$, and the kernel $H^0(P, G(x))$ is zero. This completes the proof.

COROLLARY 3.2. $\chi(\mathbf{Z}(x)) = -\mu(0, x)$.

Proof. By Lemma 3.1, $\chi(\mathbf{Z}(x)) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank}(H^i(P, \mathbf{Z}(x)) = \operatorname{rank}(H^0(P, \mathbf{Z}(x))) - \sum_{i=0}^{\infty} (-1)^i \operatorname{rank}(\tilde{H}^i((0, x), \mathbf{Z}))$. If x is minimal, the second term vanishes since $(0, x) = \varnothing$; and the first term is 1. For x minimal in P, we have $\mu(0, x) = -1$, so the result holds in this case. If x is not minimal, the first term vanishes, while the second term is one less than the Euler characteristic, $\chi(0, x)$, of the ordered set $(0, x)$ regarded as a simplicial complex. Now Rota [11, Corollary 2 of Theorem 3] has shown that $\chi(0, x) = \mu(0, x) + 1$. Therefore, the result follows also in this case.

Thus we have a cohomological interpretation of the Möbius function. This interpretation is not too far removed from that of Rota [11], the difference being the introduction of sheaf cohomology. Compare also Griffiths [10, Section 16].

COROLLARY 3.3. *$H^i(P, G(x)) = 0$ for $i >$ the maximum length of a chain in the ordered set $(0, x)$.*

In general, of course, one gets a great deal of garbage in dimensions $0 \leqslant i <$ the maximum length of a chain in $(0, x)$. There is, however, an important case where this does not occur.

A finite ordered set Q is said to be a (*finite*) *geometric lattice* if it satisfies the following conditions (see [3, Chapter 2]):

(a) Q is a lattice (hence has a minimum 0 and a maximum 1);

(b) every $x \in Q$ is a join of atoms;

(c) if $x, y \in Q$ cover $x \wedge y$, then $x \vee y$ covers both x and y.

PROPOSITION 3.4. *Let P be a lower finite ordered set such that $\hat{P}$ is a geometric lattice. Let $x \in P$, and let G be an abelian group. If n is the maximum length of a chain in $(0, x)$, then*

(a) $H^i(P, G(x)) = 0$ *for* $i < n$,

(b) $H^n(P, G(x))$ *is a direct sum of* $(-1)^{n+1} \mu(0, x)$ *copies of* G.

Proof. The result is clearly trivial for $n = 0$. We therefore assume that $n > 0$. The proof of (a) then reduces to proving that $\tilde{H}^i((0, x), G)$ is zero for $i < n - 1$, by Lemma 3.1. Part (b) reduces to showing that $\tilde{H}^{n-1}((0, x), G)$ is a direct sum of $|\mu(0, x)|$ copies of G. By the Universal Coefficient Theorem and Corollaries 3.2 and 3.3, we need only show that $\tilde{H}_i((0, x), \mathbf{Z}) = 0$ for $i < n - 1$, and that $\tilde{H}_{n-1}((0, x), \mathbf{Z})$ is a free abelian group. The result is then a consequence of a theorem of Folkman [7, Theorem 4.1].

Let Q be an ordered set with a minimum element 0. A function $r: Q \to \mathbf{Z}$ is a *rank function* for Q if it satisfies:

(a) $r(0) = 0$,

(b) if x covers y in Q, then $r(x) = r(y) + 1$.

It is easily seen that if Q is locally finite, a rank function for Q is unique when it exists. Suppose that Q is locally finite and has a rank function, and that $r^{-1}(m)$ is finite for all $m \in \mathbf{Z}$. We define the kth *Whitney number* (of the first kind) to be

$$w_k = \sum_{r(x)=k} \mu(0, x).$$

In particular, if Q is a geometric lattice, then Q has a rank function; hence the Whitney numbers are defined for Q.

We now define a sheaf $\mathscr{W}$ on P, for an arbitrary ordered set P, by $\mathscr{W}(V_x) = \mathbf{Z}$, for $x \in P$, with the restrictions being the *zero* homomorphism. When P is finite, this sheaf is just the direct sum of the sheaves $\mathbf{Z}(x)$ as x ranges over P. In a similar fashion, we define a sheaf $\mathscr{W}_G$ for any abelian group G.

THEOREM 3.5. *Let P be an ordered set such that $\hat{P}$ is a geometric lattice. Then $H^i(P, \mathscr{W})$ is a free abelian group of rank $(-1)^{i+1} w_{i+1}$ for all $i \geqslant 0$.*

Proof. Since cohomology commutes with finite direct sums, we need only add up the ranks of the groups in Proposition 3.4, being careful to keep them in the right dimension. The result then follows.

Hence the cohomology of $\mathscr{W}$ has the (absolute values of the) Whitney numbers as its set of Betti numbers.

We note that the condition on $\hat{P}$ in Theorem 3.5 cannot be easily removed. Indeed, one can find ordered sets P for which $\hat{P}$ is a finite lattice with a rank function, but for which $(-1)^k w_k$ is negative for some k. In general, one can only assert the following. Let P be an ordered set for which the Whitney numbers are defined. Define the sheaf $\mathscr{W}_i$ on P by

$$\mathscr{W}_i(V_x) = \begin{cases} \mathbf{Z} & \text{if } r(x) = i, \\ 0 & \text{if } r(x) \neq i, \end{cases}$$

for all $x \in P$, with the restrictions being the zero homomorphisms. Then for all $i > 0$ we have $\chi(\mathscr{W}_i) = -w_i$. These sheaves appear in a more natural setting in the spectral sequence to be defined in the next section.

4. A Spectral Sequence

Let P be an ordered set. An *(increasing) filtration* on P is a sequence $\{P^s\}_{s\in\mathbf{Z}}$ of closed subsets of P such that $P^s \subseteq P^{s+1}$ for all s and such that $\bigcap_{s\in\mathbf{Z}} P^s = \varnothing$.

Let Q be any subset of P and $\mathscr{F}$ any sheaf on P. We define a sheaf $\mathscr{F}_Q$ on P by

$$\mathscr{F}_Q(V_x) = \begin{cases} \mathscr{F}(V_x) & \text{if } x \in Q, \\ 0 & \text{if } x \notin Q, \end{cases}$$

with the restrictions being the zero homomorphism unless the entire interval $[x, y]$ is in Q in which case the restriction $\mathscr{F}_Q(V_x) \to \mathscr{F}_Q(V_y)$ is the same as the corresponding restriction $\mathscr{F}(V_x) \to \mathscr{F}(V_y)$ in $\mathscr{F}$. In particular, if P is given a filtration as above, we write $\mathscr{F}^s$ for $\mathscr{F}_{P^s}$ and $\mathscr{F}_s$ for $\mathscr{F}_{P-P^s}$.

For an open subset U of P, the inclusion $\mathscr{F}_U \to \mathscr{F}$ is a morphism of sheaves. This will not generally hold for an arbitrary subset of P. In particular, if P is given a filtration, we have a short exact sequence of sheaves on P for all $s \in \mathbf{Z}$:

$$0 \to \mathscr{F}_s \to \mathscr{F} \to \mathscr{F}^s \to 0$$

which does not split in general. We also have an inclusion of sheaves $\mathscr{F}_{s+1} \to \mathscr{F}_s$ for all s. Hence we may speak of the sheaf $\mathscr{F}_s/\mathscr{F}_{s+1}$ on P whose support is contained in $P^{s+1} - P^s$.

Theorem 4.1. *Let P be an ordered set with a filtration. Let $\mathscr{F}$ be any sheaf on P. Then there is an E_1-spectral sequence:*

$$H^{p+q}(P, \mathscr{F}_p/\mathscr{F}_{p+1}) \Rightarrow H^n(P, \mathscr{F}).$$

Proof. This is a pretty standard result in homological algebra. See, for example, Cartan–Eilenberg [2, XV.7]. In the notation developed there, we set $H(p, q) = H^*(P, \mathscr{F}_p/\mathscr{F}_q)$ for all pairs (p, q) such that $-\infty \leqslant p \leqslant q \leqslant +\infty$, where $\mathscr{F}_{-\infty} = \mathscr{F}$ and $\mathscr{F}_\infty = 0$. Axiom (SP5) is a consequence of the fact that cohomology commutes with direct limits. The E_1-differentials are the connecting homomorphisms of the short exact sequences:

$$0 \to \mathscr{F}_{p+1}/\mathscr{F}_{p+2} \to \mathscr{F}_p/\mathscr{F}_{p+2} \to \mathscr{F}_p/\mathscr{F}_{p+1} \to 0.$$

The abutment of the spectral sequence has a filtration given by

$$\begin{aligned} H^n(P, \mathscr{F})_p &= \operatorname{Im}(H^n(P, \mathscr{F}_p) \to H^n(P, \mathscr{F})) \\ &= \operatorname{Ker}(H^n(P, \mathscr{F}) \to H^n(P, \mathscr{F}^p)). \end{aligned}$$

The most interesting special case of Theorem 4.1 is that in which P is an ordered set for which the Whitney numbers are defined and for which $P^s = \{x \in P \mid r(x) < s\}$. Then $P^{s+1} - P^s$ is the set of elements of P of rank s. Since $P^{s+1} - P^s$ is an antichain, i.e., a totally unordered subset of P, the sheaf $\mathscr{F}_s/\mathscr{F}_{s+1}$ has a particularly simple form; namely, it is isomorphic to the direct sum of the sheaves $(\mathscr{F}(V_x))(x)$ as x ranges over the elements of P of rank s. The following result is then an immediate consequence of Lemma 3.1, Theorem 4.1 and the remarks at the end of Section 3.

COROLLARY 4.2. *Let P be an ordered set for which the Whitney numbers are defined. Let $\mathscr{F}$ be any sheaf on P. Then there is a fourth quadrant spectral sequence with*

$$E_1^{p,q} = \begin{cases} \sum_{r(x)=p} \tilde{H}^{p+q-1}((0, x), \mathscr{F}(V_x)) & \text{for } p > 1, \\ \sum_{r(x)=1} \mathscr{F}(V_x) \quad \text{for } p = 1 & \text{and} \quad q = -1, \\ 0 \quad \text{in all other cases,} \end{cases}$$

such that $E_1^{p,q} \Rightarrow H^n(P, \mathscr{F})$. Moreover, if $\mathscr{F}(V_x) \cong \mathbf{Z}$ for all $x \in P$, then for $p > 0$ we have $\chi(E_1^{p,*}) = w_p$.

Now in the special case of an ordered set P for which $\hat{P}$ is a geometric lattice, we can use Proposition 3.4 to give an explicit computation of the groups in Corollary 4.2.

COROLLARY 4.3. *Let P be an ordered set for which $\hat{P}$ is a geometric lattice. Let $\mathscr{F}$ be a sheaf on P. Then there is a fourth quadrant spectral sequence with*

$$E_1^{p,q} = \begin{cases} \sum_{r(x)=p} \mathscr{F}(V_x)^{\oplus(-1)^p \mu(0,x)} & \text{if } q = -1 \quad \text{and} \quad p \geqslant 1, \\ 0 & \text{in all other cases,} \end{cases}$$

such that $E_1^{p,q} \Rightarrow H^n(P, \mathscr{F})$. Moreover, the spectral sequence degenerates at the E_2-term with $H^{p-1}(P, \mathscr{F}) \cong E_2^{p,-1}$ for $p \geqslant 1$.

The spectral sequence in Corollary 4.3, therefore, has a particularly simple form: $E_2^{p,-1}$ is the pth cohomology group of the complex $E_1^{*,-1}$ at which point the spectral sequence degenerates. The differentials of the complex $E_1^{*,-1}$ are given by the connecting homomorphisms of the short exact sequences in the proof of Theorem 4.1. When these sequences split, e.g., when $\mathscr{F} = \mathscr{W}_G$, these differentials vanish, and we have $E_1^{p,-1} \cong E_2^{p,-1} \cong H^{p-1}(P, \mathscr{F})$. The following result is then immediate.

THEOREM 4.4. *Let P be an ordered set for which $\hat{P}$ is a geometric lattice. Let $\mathscr{F}$ be a sheaf on P all of whose stalks are isomorphic to the same abelian group G. Then there is a structure of a complex on $H^*(P, \mathscr{W}_G)$ such that its cohomology is $H^*(P, \mathscr{F})$.*

5. STANDARD RESOLUTIONS

Let $\mathscr{F}$ be a sheaf on the ordered set P. The *support* of $\mathscr{F}$ is the subset $|\mathscr{F}| = \{x \in P \mid \mathscr{F}(V_x) \neq 0\}$ of P. Suppose also that P is locally finite. An element f of $I(P)$ is said to be *transitive* if, for $x \leqslant y \leqslant z$ in P, it satisfies

$$f(x, y) f(y, z) = f(x, z).$$

Suppose that all the stalks of $\mathscr{F}$ are free abelian groups of ranks 0 or 1. If we fix a choice of isomorphism of each stalk $\mathscr{F}(V_x)$ with $\mathbf{Z}$, for $x \in |\mathscr{F}|$, then $\mathscr{F}$ defines a transitive element of $I(P)$ as follows. For $x \leqslant y$ in $|\mathscr{F}|$, $f(x, y)$ is the image of $1 \in \mathscr{F}(V_x)$ under the restriction $\mathscr{F}(V_x) \to \mathscr{F}(V_y)$. For $x \leqslant y$ in P and either $x \notin |\mathscr{F}|$ or $y \notin |\mathscr{F}|$, we define $f(x, y) = 0$. Conversely, every transitive element of $I(P)$ defines such a sheaf on P.

Now, if we allow the choice of isomorphisms of the stalks of $\mathscr{F}$ with $\mathbf{Z}$, on the support of $\mathscr{F}$, to vary, then the element of $I(P)$ defined above may change. Let H be the subgroup of $I(P)$ consisting of invertible diagonal elements, i.e., consisting of the elements $h \in I(P)$ for which $h(x, x) = \pm 1$ for $x \in P$ and for which $h(x, y) = 0$ for $x < y$ in P. Then the conjugacy class of f by elements of H is, nevertheless, uniquely determined by the sheaf $\mathscr{F}$; and, conversely, such a conjugacy class determines the sheaf $\mathscr{F}$ uniquely, up to isomorphism. The sheaf $\mathscr{W}$ corresponds to the identity element δ of $I(P)$, while the sheaf $\tilde{\mathbf{Z}}$ of locally constant functions on P corresponds to the zeta function $\zeta \in I(P)$.

Let $\mathscr{F}$ be a sheaf on P as above with f a corresponding transitive element of $I(P)$. Let $C_*(P, f)$ be a chain complex defined as follows. $C_n(P, f)$ is the subgroup of $C_n(P, \mathbf{Z})$ generated by n-simplices of P whose maximum element lies in $|\mathscr{F}|$. The differential d_f: $C_n(P, f) \to C_{n-1}(P, f)$ is defined by

$$d_f(a_0 < \cdots < a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 < \cdots < \hat{a}_i < \cdots < a_n)$$
$$+ (-1)^n f(a_{n-1}, a_n)(a_0 < \cdots < a_{n-1}),$$

for each n-simplex $(a_0 < \cdots < a_n) \in C_n(P, f)$. The $\mathbf{Z}$-dual of this complex is denoted $C^*(P, f)$. The differential on $C^n(P, f)$ is then given by

$$\begin{aligned} &d_f(\alpha)(a_0 < \cdots < a_{n+1}) \\ &= \sum_{i=0}^{n} (-1)^i \alpha(a_0 < \cdots < \hat{a}_i < \cdots < a_{n+1}) \\ &+ (-1)^{n+1} f(a_n, a_{n+1})\, \alpha(a_0 < \cdots < a_n), \end{aligned}$$

for $\alpha \in C^n(P, f)$ and $(a_0 < \cdots < a_{n+1})$ an $(n+1)$-simplex in $C_{n+1}(P, f)$.

PROPOSITION 5.1. *Let P be a locally finite ordered set, and let $\mathscr{F}$ be a sheaf on P all of whose stalks are free abelian groups of rank* 0 *or* 1. *Let $f \in I(P)$ be a transitive element which corresponds to $\mathscr{F}$. Then*

$$H^*(P, \mathscr{F}) \cong H^*(C^*(P, f)).$$

Proof. Deheuvels [5, Section 10].

For example, when $\mathscr{F} = \mathscr{W}$, $H^*(P, \mathscr{W})$ is the cohomology of the cochain complex $C^*(P, \delta)$. Explicitly, $C^*(P, \delta)$ is the same as $C^*(P, \mathbf{Z})$ as a group, but the differential in $C^*(P, \delta)$ is

$$d_\delta(\alpha)(a_0 < \cdots < a_{n+1})$$

$$= \sum_{i=0}^{n} (-1)^i \alpha(a_0 < \cdots < \hat{a}_i < \cdots < a_{n+1}),$$

for $\alpha \in C^n(P, \mathbf{Z})$ and for each $(n+1)$-simplex $(a_0 < \cdots < a_{n+1})$ in $C_{n+1}(P, \mathbf{Z})$.

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